

# Parameter Estimation and Inference with Spatial Lags and Cointegration\*

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February 15, 2013

## Abstract

This article studies dynamic panel data models where the long run outcome for a particular cross-section is affected by a weighted average of the outcomes in the other cross-sections. We show that imposing such a structure implies a model with several cointegrating relationships that, unlike in the standard case, are nonlinear in the coefficients to be estimated. Assuming that the weights are exogenously given, we extend the dynamic ordinary least squares methodology and provide a dynamic two-stage least squares estimator. We derive the large sample properties of our proposed estimator under a set of low-level assumptions and investigate its small sample distribution in a simulation study. Then our methodology is applied to US financial market data, which consist of credit default swap spreads, firm specific and industry data. We construct the economic space by using a "closeness" measure for firms based on input-output matrices. This measure reflects the spreading or contagion of credit risk through the supply chain. Our estimates show that this particular form of spatial correlation of credit default spreads is substantial and highly significant.

**Keywords:** Dynamic Ordinary Least Squares, Cointegration, Credit Risk, Credit Default Swaps, Spatial Autocorrelation.

**JEL:** C31, C32.

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\*The authors thank Manfred Deistler, Benedikt Pötscher, Martin Wagner as well the participants of the *CFE 2012 Conference*, the *IHS Econometrics Research Seminar* and *EBS Research Seminar* for interesting discussions and comments.

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# 1 Introduction

This article investigates the estimation of non-standard cointegrating relationships under the presence of regressor endogeneity and serial correlation in the disturbances. Following literature on panel cointegration we augment the cointegrating vectors by peer or neighborhood effects, which are modeled as *spatial lags* following Cliff and Ord (1973). We observe that the inclusion of a spatial lag into an equation for the levels of non-stationary variables is a further source of regressor endogeneity. Since existing estimation procedures do not cope with this kind of endogeneity, we propose to use a dynamic two-stage least squares (*D2SLS*) estimator, which combines dynamic least squares (*DOLS*) and two stage least squares (*2SLS*) estimation. In addition to the serial lags used by *DOLS*, our estimation procedure uses cross-sectional (or spatial) lags of the regressors as instruments to control for the endogeneity of the spatial lags in the cointegrating vectors.

Estimating the parameters of a cointegrating relationship can be perceived to be a relatively straightforward task. The challenges arise when we attempt to provide guidance for statistical inference and hypothesis testing. Two broad approaches have emerged in the literature. One possibility is to use a simple estimation routine, e.g. ordinary least squares (*OLS*) and then work out the (sometimes complicated) large sample distribution of the estimated parameters, e.g. Phillips and Hansen (1990), Phillips and Loretan (1991). Another opportunity is to adjust the estimation routine, such that the large sample distribution is either simpler, or free of nuisance parameters. Examples along these lines are the *fully modified least squares* estimator (see e.g. Phillips and Hansen (1990), Phillips and Moon (1999), Pedroni (2000); or Vogelsang and Wagner (2011) for a fixed-b perspective) and the *dynamic least squares* (*DOLS*) approach. Dynamic least squares estimation includes time-series leads and lags of the regressors to control for the serial correlation and regressor endogeneity. This kind of estimator has been proposed by Phillips and Loretan (1991), Saikkonen (1991) and Stock and Watson (1993). It has been applied to panel data e.g. in Kao and Chiang (2000), Mark and Sul (2003) and Mark et al. (2005). The *DOLS* approach will also be used and augmented in this article.

In the rest of the paper we first describe our model and the formal assumptions in Section 2. Section 3 describes the *D2SLS* estimation and states our large sample results. We then investigate the small

sample properties of the *D2SLS* estimator in Section 4 and provide an illustrative application to modeling correlation of credit default swaps in Section 5. Finally, Section 6 offers conclusions.

## 2 The Model

Suppose that the data are generated from

$$y_{it} = \rho \sum_{j=1}^n W_{ij} y_{jt} + \beta' \mathbf{x}_{it} + \alpha_i + u_{it}^\dagger = \rho y_{it}^* + \beta' \mathbf{x}_{it} + \alpha_i + u_{it}^\dagger, \quad (1)$$

where  $y_{it}$  is the scalar response random variable and  $\mathbf{x}_{it}$  is a  $k \times 1$  vector of prediction random variables.  $t = 1, \dots, T$  is the time index,  $i = 1, \dots, n$  is the cross-sectional dimension.  $n$  is kept fixed throughout the following analysis. The term  $y_{it}^* = \sum_{j=1}^n W_{ij} y_{jt}$  is referred to as a *spatial lag* (see e.g. Cliff and Ord (1973), Kelejian and Prucha (1998) or Kelejian and Prucha (1999)) and represents the long-run impact of the neighboring observations on  $y_{it}$ . We collect the weights  $W_{ij}$  into an  $n \times n$  spatial weights matrix  $\mathbf{W}$ . The number of spatial lags in (1) is  $k_\rho = 1$ . This restriction is used to keep the notation simple. The theory considered in this article can also be applied to a model where  $y_{it} = \rho_1 \sum_{j=1}^n W_{1,ij} y_{jt} + \dots + \rho_{k_\rho} \sum_{j=1}^n W_{k_\rho,ij} y_{jt} + \dots$  in a straightforward way. We follow the spatial econometrics literature and maintain the following assumptions regarding the cross-sectional (or spatial) structure of the model:

**Assumption 1.** [Spatial Lag] The spatial weights  $W_{ij}$  are non-stochastic and observable with  $W_{ii} = 0$  and  $\mathbf{W} \neq \mathbf{0}_{n \times n}$ . The parameter  $\rho$  is such that largest absolute eigenvalue of  $\rho \mathbf{W}$  is smaller than one.

The restriction that  $W_{ii} = 0$  is a normalization of the model, which requires that no observation is its own neighbor. The second part of the assumption guarantees that the matrix  $(\mathbf{I}_n - \rho \mathbf{W})$  is invertible (see e.g. Corollary 5.6.16 in Horn and Johnson (1985));  $\mathbf{I}_n$  stands for the identity matrix of dimension  $n$ .<sup>1</sup> The invertibility of the matrix  $\mathbf{I}_n - \rho \mathbf{W}$  is needed in order to provide a unique solution of the model and rule out multiple solutions for  $y_{it}$  that would be consistent with the explanatory variables and disturbances.

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<sup>1</sup>The requirement for invertibility is that there exists some matrix norm  $\|\cdot\|$  such that  $\|\rho \mathbf{W}\| < 1$ . Given that the spectral radius is the smallest matrix norm (cf. Theorem 5.6.9 in Horn and Johnson (1985)), our assumption is in this sense least restrictive. It will, for example, be satisfied when the maximum absolute row or column sums of  $\rho \mathbf{W}$  are less than one.

The disturbance term is assumed to include an individual-specific effect  $\alpha_i$  and an idiosyncratic component  $u_{it}^\dagger$  that is independent across  $i$  but possibly dependent across  $t$ . Analogously to Mark and Sul (2003) the prediction random variable  $\mathbf{x}_{it}$  is assumed to be integrated of order one,  $I(1)$ , and to be generated from

$$\Delta \mathbf{x}_{it} = \mathbf{v}_{it}. \quad (2)$$

In order to fully specify the model, we augment our set of assumptions by defining the process generating the disturbances:

**Assumption 2.** [Error Dynamics I; see Mark and Sul (2003), Mark et al. (2005), Phillips (2006)] Let us define the stacked vector  $\mathbf{w}_{it}^\dagger = (u_{it}^\dagger, v_{it}^\dagger)'$ . Then  $(\mathbf{w}_{it}^\dagger)$  has a moving average representation

$$\mathbf{w}_{it}^\dagger = \Psi_i^\dagger(L) \varepsilon_{it}^\dagger,$$

where  $\varepsilon_{it}^\dagger$  is *iid* over both  $i$  and  $t$  with mean vector  $\mathbf{0}_{k+1}$ ,  $k+1 \times k+1$  positive definite covariance matrix  $\Sigma_{\varepsilon_i}$  and finite fourth moments.  $\Psi_i^\dagger(L) = \sum_{j=0}^{\infty} \Psi_{ij}^\dagger L^j$  is a  $k+1 \times k+1$  dimensional matrix lag polynomial in the lag operator  $L$ , with  $\sum_{j=0}^{\infty} j \left[ \Psi_{ij}^\dagger \right]_{(m,n)} < \infty$  where  $\left[ \Psi_{ij}^\dagger \right]_{(m,n)}$  is the  $m, n$  element of the matrix  $\Psi_{ij}^\dagger$ .  $\Psi_{i0}^\dagger = \mathbf{I}_{k+1}$ .

We shall denote the short-run  $k+1 \times k+1$  covariance matrix  $\Gamma_{i0}^\dagger$  and the autocovariance matrix  $\Gamma_{ij}^\dagger$  as

$$\Gamma_{i0}^\dagger = \mathbb{E} \left( \mathbf{w}_{it}^\dagger \mathbf{w}_{it}^{\dagger'} \right) \text{ and } \Gamma_{ij}^\dagger = \mathbb{E} \left( \mathbf{w}_{it}^\dagger \mathbf{w}_{i,t-j}^{\dagger'} \right). \quad (3)$$

We will also use the following notation:  $\Gamma_{uu,ij}^\dagger$  is the 1, 1 element of  $\Gamma_{ij}^\dagger$ ,  $\Gamma_{uv,j}^\dagger$  corresponds to  $\left[ \Gamma_{ij}^\dagger \right]_{(2:k+1,1)}$ ,  $\Gamma_{vu,j}^\dagger$  corresponds to  $\left[ \Gamma_{ij}^\dagger \right]_{(1,2:k+1)}$  while  $\Gamma_{vv,j}^\dagger$  corresponds to  $k \times k$  submatrix  $\left[ \Gamma_{ii}^\dagger \right]_{(2:k+1,2:k+1)}$  where the notation  $(a : b, c : d)$  stands for "from row  $a$  to  $b$  and column  $c$  to  $d$ ".

Let us define  $\mathbf{w}_t^\dagger = (\mathbf{w}_{1t}^{\dagger'}, \dots, \mathbf{w}_{nt}^{\dagger'})'$ ,  $\mathbf{u}_t^\dagger = (u_{1t}^\dagger, \dots, u_{nt}^\dagger)'$  and  $\mathbf{v}_t = (\mathbf{v}_{1t}', \dots, \mathbf{v}_{nt}')'$ . Then the  $(k+1) \cdot n \times (k+1) \cdot n$  covariance matrices  $\Gamma_0^\dagger = \mathbb{E} \left( \mathbf{w}_t^\dagger \mathbf{w}_t^{\dagger'} \right)$  and  $\Gamma_j^\dagger = \mathbb{E} \left( \mathbf{w}_t^\dagger \mathbf{w}_{t-j}^{\dagger'} \right)$  are block diagonal with the blocks  $\Gamma_{i0}^\dagger$  and  $\Gamma_{ij}^\dagger$  along the main diagonal ( $i = 1, \dots, n$ ). The  $k+1 \times k+1$  long run covariance matrix

$\mathbf{\Omega}_i^\dagger$  of  $\mathbf{w}_{it}^\dagger$  is given by

$$\begin{aligned}\mathbf{\Omega}_i^\dagger &= \sum_{j=-\infty}^{\infty} \mathbb{E} \left( \mathbf{w}_{it}^\dagger \mathbf{w}_{i,t+j}^{\dagger'} \right) = \mathbf{\Psi}_i^\dagger(1) \mathbf{\Sigma}_{\varepsilon_i} \mathbf{\Psi}_i^\dagger(1)' = \mathbf{\Gamma}_{i0}^\dagger + \sum_{j=1}^{\infty} \left( \mathbf{\Gamma}_{ij}^\dagger + \mathbf{\Gamma}_{ij}^{\dagger'} \right) \\ &= \begin{pmatrix} \mathbf{\Omega}_{uu,i}^\dagger & \mathbf{\Omega}_{vu,i}^\dagger \\ \mathbf{\Omega}_{uv,i}^\dagger & \mathbf{\Omega}_{vv,i}^\dagger \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_{uu,i0}^\dagger & \mathbf{\Gamma}_{vu,i0}^\dagger \\ \mathbf{\Gamma}_{uv,i0}^\dagger & \mathbf{\Gamma}_{vv,i0}^\dagger \end{pmatrix} + \sum_{j=1}^{\infty} \left[ \begin{pmatrix} \mathbf{\Gamma}_{uu,ij}^\dagger & \mathbf{\Gamma}_{vu,ij}^\dagger \\ \mathbf{\Gamma}_{uv,ij}^\dagger & \mathbf{\Gamma}_{vv,ij}^\dagger \end{pmatrix} + \begin{pmatrix} \mathbf{\Gamma}_{uu,ij}^\dagger & \mathbf{\Gamma}_{vu,ij}^\dagger \\ \mathbf{\Gamma}_{uv,ij}^\dagger & \mathbf{\Gamma}_{vv,ij}^\dagger \end{pmatrix}' \right].\end{aligned}\quad (4)$$

The long-run covariance matrix of  $\mathbf{w}_i^\dagger$ , denoted as  $\mathbf{\Omega}^\dagger$ , is then also block diagonal with the blocks  $\mathbf{\Omega}_i^\dagger$  along the main diagonal. Analogously, the matrices  $\mathbf{\Omega}_{uu}^\dagger$  and  $\mathbf{\Omega}_{vv}^\dagger$  contain scalars  $\mathbf{\Omega}_{uu,i}^\dagger$  and  $k \times k$  blocks  $\mathbf{\Omega}_{vv,i}^\dagger$  along their main diagonal.

Given the covariance structure, we want to exclude a cointegration relationship between the terms of  $\mathbf{x}_{it}$ . Therefore we impose the following assumption:

**Assumption 3.** [Error Dynamics II; see Phillips (2006)]

$\mathbf{\Psi}_i^\dagger(1)$  is non-singular such that  $\mathbf{x}_{it}$  is a full rank integrated process and  $\mathbf{\Omega}_{vv,i}^\dagger$  has full rank  $k$ .

By the independence across  $i$  assumption (i.e. Assumption 2), this implies that the rank of  $\mathbf{\Omega}_{vv}$  is  $nk$ .

In the next step we follow *DOLS* literature and remove the serial correlation by projecting on the leads and lags of  $\Delta \mathbf{x}_{it}$ . Note that Assumption 3 implies that potentially all leads and lags of  $\Delta \mathbf{x}_{it}$  are correlated with  $u_{it}^\dagger$ . For each sample size, *DOLS* estimation uses a finite number leads of lags, abbreviated by  $p$  in the following, to control for this correlation. Using such a truncation scheme will result in a specific truncation error  $\mathbf{e}_{it}$ . However, under the conditions provided in Saikkonen (1991) this error will disappear asymptotically. In more detail, the projection of  $u_{it}^\dagger$  on the  $p$  leads and lags of  $\Delta \mathbf{x}_{it}$  yields a truncation component, a truncation error  $\mathbf{e}_{it}$  plus a *new* disturbance  $u_{it}$ :

$$u_{it}^\dagger = \sum_{s=-p}^{+p} \delta'_{i,s} \Delta \mathbf{x}_{i,t-s} + \sum_{s>p, s<-p} \delta'_{i,s} \Delta \mathbf{x}_{i,t-s} + u_{it} = \delta'_i \zeta_{it} + \mathbf{e}_{it} + u_{it} = \delta'_i \zeta_{it} + \underline{u}_{it}. \quad (5)$$

$\Delta \mathbf{x}_{i,t-s}$  and  $\delta_{i,s}$  are a vectors of dimension  $k$ , such that the  $(2p+1)k \times 1$  dimensional vectors of projection

variables and projection coefficients are given by

$$\zeta_{it} = (\Delta \mathbf{x}'_{i,t-p}, \dots, \Delta \mathbf{x}'_{i,t}, \dots, \Delta \mathbf{x}'_{i,t+p})' = (\mathbf{v}'_{i,t-p}, \dots, \mathbf{v}'_{i,t}, \dots, \mathbf{v}'_{i,t+p})' \text{ and } \delta_i = (\delta'_{i,-p}, \dots, \delta'_{i,+p})'. \quad (6)$$

$\zeta_{it}$  is by construction orthogonal to the noise term  $u_{it}$ . The term  $\underline{u}_{it} = \mathbf{e}_{it} + u_{it}$  can still be correlated with  $\Delta \mathbf{x}_{it}$  for some  $p < \infty$ . Now we impose an additional restriction on the error dynamics that will guarantee that the truncation error  $\mathbf{e}_{it}$  converges to zero:

**Assumption 4.** [Error Dynamics III; see Saikkonen (1991), Mark et al. (2005)]

Given the Assumptions 2 and 3. Suppose that  $p = p(T)$ .  $p(T)$  fulfills  $\frac{p(T)^3}{T} \rightarrow 0$  and  $\sqrt{T} \sum_{|s|>p(T)} \|\delta_{i,s}\|_2 \rightarrow 0$  as  $T \rightarrow \infty$ .

$\|\cdot\|_2$  stands for the Euclidian norm. Assumption 4 requires that  $p(T)$  does not grow too fast, while the second term restricts the dependence between the noise term and the regressor. Based on Assumptions 2 to 4 and equation (5), if  $T$  becomes large then – due to the increase in the number of leads and lags  $p(T)$  – the truncation error  $\mathbf{e}_{it}$  becomes small. I.e. the difference between  $\underline{u}_{it}$  and  $u_{it}$  becomes small, such that also  $\underline{u}_{it}$  becomes orthogonal to  $\zeta_{it}$  for  $T \rightarrow \infty$ .<sup>2</sup> Hence we arrive at the *new* covariance stationary process  $(\mathbf{w}_{it}) = (u_{it}, \mathbf{v}'_{it})' = \Psi_i(L) \varepsilon_{it}$  which has mean zero, covariance matrix  $\mathbf{\Gamma}_{i0}$  and autocovariance  $\mathbf{\Gamma}_{ij}$ . These matrices are of the structure

$$\mathbf{\Gamma}_{ij} = \mathbb{E}(\mathbf{w}_{it} \mathbf{w}'_{i,t-j}) = \begin{pmatrix} \Gamma_{uu,ij} & \mathbf{0}_{1 \times k} \\ \mathbf{0}_{k \times 1} & \mathbf{\Gamma}_{vv,ij} \end{pmatrix}, \quad (7)$$

where  $\Gamma_{uu,ij} = \mathbb{E}(u_{it} u_{i,t-j})$ ,  $\mathbf{\Gamma}_{vv,ij} = \mathbb{E}(\mathbf{v}_{it} \mathbf{v}'_{i,t-j})$  and  $j \in \mathbb{Z}$ .

In addition our model includes a full set of individual specific effects and hence a set of individual dummies  $\alpha_i$  has been included to the regression (1) (*fixed effects specification*). In order to simplify the algebra, we shall use the within transformation and derive the asymptotic distribution of the estimates of the slope coefficients  $\rho$  and  $\beta$  using within-transformed data. In a linear regression, these estimated slope coefficients are algebraically equivalent to the LSDV estimates (see e.g. Baltagi (2008)[p. 11]). Here it is

<sup>2</sup>For a short discussion on the truncation error and the Assumption 4 we refer the reader to Saikkonen (1991) and to Lütkepohl (2006)[Remark 1, p. 533]. For more technical details see Saikkonen (1991)[Theorem 4.1/Lemma A.5].

important to note that while the time index  $t$  goes from 1 to  $T$  in (1), while after the projection facility is applied only the observations  $p + 1, \dots, T - p$  can be used. We still use  $t$  as the index for the time period which now runs from 1 to  $T_\star$ , where  $T_\star = T - 2p$ . The variables in deviations from their individual means are

$$\begin{aligned}\tilde{y}_{it} &= y_{it} - \frac{1}{T_\star} \sum_{t=1}^{T_\star} y_{it}, \quad \tilde{\mathbf{x}}_{it} = \mathbf{x}_{it} - \frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{x}_{it}, \quad \tilde{y}_{it}^* = \sum_{j=1}^n W_{ij} \tilde{y}_{it} \\ \tilde{\zeta}_{it} &= \zeta_{it} - \frac{1}{T_\star} \sum_{t=1}^{T_\star} \zeta_{it},\end{aligned}\tag{8}$$

such that (1) after applying the within transform and the projection facility reads as follows:

$$\begin{aligned}\tilde{y}_{it} &= \rho \sum_{j=1}^n W_{ij} \tilde{y}_{jt} + \beta' \tilde{\mathbf{x}}_{it} + \tilde{u}_{it}^\dagger = \rho \tilde{\mathbf{y}}_{it}^* + \beta' \tilde{\mathbf{x}}_{it} + \tilde{u}_{it}^\dagger \\ &= \rho \sum_{j=1}^n W_{ij} \tilde{y}_{jt} + \beta' \tilde{\mathbf{x}}_{it} + \delta'_i \tilde{\zeta}_{it} + \tilde{u}_{it} = \rho \tilde{\mathbf{y}}_{it}^* + \beta' \tilde{\mathbf{x}}_{it} + \delta'_i \tilde{\zeta}_{it} + \tilde{u}_{it}.\end{aligned}\tag{9}$$

Given the assumptions on the error dynamics, the *functional central theorem* (see e.g. Johansen (1995)[Appendix], Karatzas and Shreve (1991)[Chapter 4] or Davidson (1994)[Chapters 27-30]) can be applied. If  $T_\star \rightarrow \infty$  then

$$\frac{1}{\sqrt{T_\star}} \sum_{t=1}^{[T_\star r]} \mathbf{w}_{it} \xrightarrow{d} \mathcal{B}_i(r) = \mathbf{\Omega}_i^{1/2} \mathcal{W}_i(r), \quad \text{where } r \in [0, 1].\tag{10}$$

$\xrightarrow{d}$  stands for *weak convergence / convergence in distribution*.  $\mathcal{B}_i(r) = (\mathcal{B}_{ui}(r), \mathcal{B}_{vi}(r)')'$ , where  $\mathcal{B}_{ui}$  and  $\mathcal{B}_{vi}$  are independent Brownian motions, in  $\mathbb{R}$  and  $\mathbb{R}^k$ , respectively. While  $\mathcal{B}_i$  stands for a Brownian motion with covariance matrix  $\mathbf{\Omega}_i$ ,  $\mathcal{W}_i$  stands for a standard Brownian motion, where  $\mathcal{W}_i(r) = (\mathcal{W}_{ui}(r), \mathcal{W}_{vi}(r)')'$ .  $[T_\star r]$  denotes the integer part of  $T_\star r$ .<sup>3</sup>  $\mathbf{\Omega}_i$  is the  $k + 1 \times k + 1$  long-run variance-covariance matrix of  $\mathbf{w}_{it}$ .

<sup>3</sup>In some of the following expressions we omit the borders of integration as well as the dependence of the Brownian motion on  $r$ , i.e. we write  $\int W$  instead of  $\int_0^1 W(s)ds$ , while  $\int_0^1 W(s)dW(s)$  is abbreviated by  $\int WdW$ .

Due to the independence of  $\mathcal{B}_{ui}(t)$  and  $\mathcal{B}_{vi}(t)$ , this matrix is of the structure

$$\mathbf{\Omega}_i = \begin{pmatrix} \Omega_{uu,i} & \mathbf{0}_{1 \times k} \\ \mathbf{0}_{k \times 1} & \Omega_{vv,i} \end{pmatrix} = \mathbf{\Gamma}_{i0} + \sum_{j=1}^{\infty} (\mathbf{\Gamma}_{ij} + \mathbf{\Gamma}'_{ij}), \quad (11)$$

where the matrices  $\mathbf{\Gamma}_{ij}$  are given by (7). For the demeaned term  $\tilde{v}_{it} = v_{it} - \frac{1}{T_\star} \sum_{t=1}^{T_\star} v_{it}$  we get  $\frac{1}{\sqrt{T_\star}} \sum_{t=1}^{[T_\star r]} \tilde{v}_{it} = \frac{1}{\sqrt{T_\star}} \sum_{t=1}^{[T_\star r]} \left( v_{it} - \frac{1}{T_\star} \sum_{t=1}^{T_\star} v_{it} \right) \xrightarrow{d} \mathcal{B}_{vi}(r) - r\mathcal{B}_{vi}(1)$ .  $\mathcal{B}_{vi}(r) - r\mathcal{B}_{vi}(1)$  is a Brownian bridge. Since  $\mathbf{x}_{it}$  is an  $I(1)$  process,  $\mathbf{x}_{it}$  arises from a partial sum process. Then  $\tilde{\mathbf{x}}_{it} = \sum_{\iota=1}^t v_{i\iota} - \frac{1}{T_\star} \sum_{t=1}^{T_\star} \sum_{\iota=1}^t v_{i\iota}$ . By the continuous mapping theorem (see Klenke (2008)[p. 257], Davidson (1994)[Theorem 26.13 & 30.2]) the  $T_\star \rightarrow \infty$  limit is given by

$$\frac{1}{\sqrt{T_\star}} \tilde{\mathbf{x}}_{it} \xrightarrow{d} \mathcal{B}_{vi}(r) - \int_0^1 \mathcal{B}_{vi}(s) ds.$$

$\mathcal{B}_{vi}(r) - \int_0^1 \mathcal{B}_{vi}(s) ds$  will be abbreviated by  $\tilde{\mathcal{B}}_{vi}(r)$ . Last but not least Davidson (1994)[Theorem 30.2] shows that  $\frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \tilde{\mathbf{x}}_{it} \tilde{\mathbf{x}}'_{it}$  converges in distribution to  $\int_0^1 \mathcal{B}_{vi}(s) \mathcal{B}'_{vi}(s) ds$ . In addition Davidson (1994)[Theorem 30.13] and some algebra shows that  $\frac{1}{T_\star} \sum_{t=1}^{T_\star} \tilde{\mathbf{x}}_{it} \tilde{\mathbf{u}}'_{it} \xrightarrow{d} \sqrt{\mathbf{\Omega}_{uu,i}} \int_0^1 \tilde{\mathcal{B}}_{vi}(r) d\mathcal{W}_{ui}(r)$ .

Before we proceed with the estimation part we would like to discuss our model for the  $n = 2$  case. Here we observe that the cointegration equations are non-linear and due to the spatial lag component an additional source of endogeneity arises.

**Remark 1.** Consider (1) for the two-dimensional case, i.e.  $n = 2$ . Due to Assumption 1 the matrix  $\mathbf{I}_2 - \rho \mathbf{W}$  has to be invertible, such that

$$\left[ \mathbf{I}_2 - \rho \begin{pmatrix} 0 & W_{12} \\ W_{21} & 0 \end{pmatrix} \right]^{-1} = \frac{1}{1 + \rho^2 W_{12} W_{21}} \cdot \begin{pmatrix} 1 & -\rho W_{21} \\ -\rho W_{12} & 1 \end{pmatrix}. \quad (12)$$

Equations (1) and (12) now result in

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \frac{1}{1 + \rho^2 W_{12} W_{21}} \cdot \begin{pmatrix} \beta^\top \mathbf{x}_{1t} & -\rho W_{21} \beta^\top \mathbf{x}_{2t} & +u_{1t}^\dagger & -\rho W_{21} u_{2t}^\dagger & +\alpha_1 \\ -\rho W_{12} \beta^\top \mathbf{x}_{1t} & +\beta^\top \mathbf{x}_{2t} & +u_{2t}^\dagger & -\rho W_{12} u_{1t}^\dagger & +\alpha_1 \end{pmatrix}. \quad (13)$$

Equation (13) also shows the  $n = 2$  cointegration equations. The cointegrating relationships do not have



the usual linear form in the sense that the solution for  $y_{it}$  is a nonlinear function of the parameter  $\rho$ .

For an arbitrary but fixed  $n \in \mathbb{N}$  we obtain  $n$  cointegration relationships. This can be seen by looking at  $\mathbf{y}_{it} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} (\beta^\top \mathbf{x}_{it} + u_{it}^\dagger)$ . By considering the terms  $(\beta^\top \mathbf{x}_{it} + u_{it}^\dagger)$ , for  $i = 1, \dots, n$ , and Assumptions 2-4 we have  $n$  cointegration relationships. Since Assumption 1 guarantees that  $\mathbf{I}_n - \rho \mathbf{W}$  has the full rank  $n$ , the cointegration space remains of dimension  $n$ .<sup>4</sup>

Summing up, when we consider the data generated by (1) we observe that: (i)  $\mathbf{x}_{it}$  and  $u_{it}^\dagger$  are correlated by the assumptions on  $\Psi_i^\dagger$  and  $\Sigma_{\varepsilon i}$ . (ii) For  $\rho \neq 0$ ,  $y_{jt}$  depends on  $y_{it}$  and vice versa. (iii)  $u_{it}^\dagger$  and  $u_{jt}^\dagger$  are independent by Assumption 2. (iv) Since  $y_{jt}$  depends on  $y_{it}$  we know that  $\rho W_{ij} y_{jt}$  and  $u_{it}^\dagger$  have to be correlated. (Also for the within transformed data, the same correlation structure is observed.)

In the following section we shall construct an estimator where we account for "serial" endogeneity by means of the DOLS projection facility. In addition endogeneity enters via the spatial correlation modeled by  $\rho \mathbf{W}$ . To account for this kind of "spatial" endogeneity we follow the *2SLS* approach. Combining these concepts will provide us with an estimator which accounts for both sources of endogeneity.

### 3 Estimation Procedure and Large Sample Results

The goal of the following analysis is to construct the *D2SLS* estimator and show that it leads to consistent estimates of the parameters  $\rho$  and  $\beta$ . Then we provide the large sample distribution of the *D2SLS* estimator. The parameters  $\delta$  will be shown to be *nuisance parameters*. In order to write down our estimator in a compact way, we first define the model in a stacked notation. For notational simplicity we drop the tilde notation in the stacked model and define

$$\begin{aligned}
 \mathbf{y} &= (\tilde{y}_{11}, \dots, \tilde{y}_{1T_\star}, \dots, \tilde{y}_{n1}, \dots, \tilde{y}_{nT_\star})', \\
 \mathbf{y}^* &= (\tilde{y}_{11}^*, \dots, \tilde{y}_{1T_\star}^*, \dots, \tilde{y}_{n1}^*, \dots, \tilde{y}_{nT_\star}^*)', \\
 \mathbf{x} &= (\tilde{\mathbf{x}}'_{11}, \dots, \tilde{\mathbf{x}}'_{1T_\star}, \dots, \tilde{\mathbf{x}}'_{n1}, \dots, \tilde{\mathbf{x}}'_{nT_\star})', \\
 \mathbf{u} &= (\tilde{u}_{11}, \dots, \tilde{u}_{1T_\star}, \dots, \tilde{u}_{n1}, \dots, \tilde{u}_{nT_\star})', \tag{14}
 \end{aligned}$$

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<sup>4</sup>Here we thank Martin Wagner for this advise.

where  $\mathbf{y}$  and  $\mathbf{y}^*$  are of dimension  $nT_\star \times 1$ , while  $\mathbf{x}$  is an  $nT_\star \times k$  matrix. Furthermore, we have

$$\zeta\delta = \begin{pmatrix} \delta'_{11}\tilde{\zeta}_{11} \\ \vdots \\ \delta'_{nT_\star}\tilde{\zeta}_{nT_\star} \end{pmatrix} = \begin{pmatrix} \tilde{\zeta}'_{11} & \mathbf{0}_{1 \times (2p+1)k} & \mathbf{0}_{1 \times (2p+1)k} \\ \vdots & & \\ \tilde{\zeta}'_{1T_\star} & \mathbf{0}_{1 \times (2p+1)k} & \mathbf{0}_{1 \times (2p+1)k} \\ \mathbf{0}_{1 \times (2p+1)k} & \tilde{\zeta}_{21} & \mathbf{0}_{1 \times (2p+1)k} \\ & & \ddots \\ \mathbf{0}_{1 \times (2p+1)k} & \mathbf{0}_{1 \times (2p+1)k} & \tilde{\zeta}_{nT_\star} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}. \quad (15)$$

$\zeta$  is a  $nT_\star \times (2p+1)k \cdot n$  matrix, while (given  $\delta_i$  of dimension  $(2p+1)k$ )  $\delta$  is of dimension  $(2p+1)k \cdot n \times 1$ .

This provides us with model (9) in stacked form:

$$\mathbf{y} = \rho\mathbf{y}^* + \mathbf{x}\beta + \zeta\delta + \mathbf{u} = (\mathbf{y}^*, \mathbf{x})\gamma + \zeta\delta + \mathbf{u} = \mathbf{X}(\gamma', \delta')' + \mathbf{u}, \quad (16)$$

where  $\gamma = (\rho, \beta')'$ . The right hand side variables are collected in  $\mathbf{X} = (\mathbf{y}^*, \mathbf{x}, \zeta)$ .

We shall estimate the model by using instruments for the endogenous variable  $\tilde{y}_{it}^* = \sum_{j=1}^n W_{ij}\tilde{y}_{jt}$ . Here, we could proceed in an abstract way by assuming that  $q_\rho \geq k_\rho = 1$  instruments are available fulfilling the properties necessary for instrumental variable estimation (see e.g. Kitamura and Phillips (1997)). In contrast to this, we follow Kapoor et al. (2007) and base the instruments on the spatial lags of the explanatory variables. Observe that our model can be solved as

$$\mathbf{y} = [\mathbf{I}_T \otimes (\mathbf{I}_n - \rho\mathbf{W})^{-1}] (\mathbf{x}\beta + \zeta\delta + \mathbf{u}), \quad (17)$$

where the inverse exists by our Assumption 1. The matrix  $(\mathbf{I}_n - \rho\mathbf{W})^{-1}$  can then be expanded as (see e.g. Corollary 5.6.16 in Horn and Johnson (1985)):

$$(\mathbf{I}_n - \rho\mathbf{W})^{-1} = \sum_{s=0}^{\infty} (\rho\mathbf{W})^s. \quad (18)$$

This implies that variables of the form  $\sum_{j=1}^n W_{ij}\tilde{x}_{jtv}, \sum_{j=1}^n W_{ij}^2\tilde{x}_{jtv}, \dots$  are suitable instruments for  $\mathbf{W}\mathbf{y}$ .

$\tilde{x}_{jtv}$  is the element  $v$  of  $\tilde{\mathbf{x}}_{jt}$ . Note that these instruments have an intuitive interpretation: we instrument the average of the values of  $\tilde{y}_{it}$  of the neighbors/peers by the average characteristics of the neighbors (their  $\tilde{\mathbf{x}}_{it}$  values). The higher order spatial lags as instruments then use the characteristics of the neighbors of the neighbors, etc. We hence assume that the following set of instruments is used:

**Assumption 5.** [Valid Instruments] The instruments are  $\tilde{x}_{itv}^* = \sum_{j=1}^n W_{ij}^{\tau_v} \tilde{x}_{jtv}$ , where  $v = 1, \dots, q_\rho$  and  $\tau_v \in \mathbb{N}$ .  $\tilde{\mathbf{x}}_{it}^* = \left( \tilde{x}_{it1}^*, \dots, \tilde{x}_{itq_\rho}^* \right)'$  is a vector of dimension  $q_\rho$ . We assume that these instruments fulfill the requirements for instrumental variable estimation as stated e.g. in Ruud (2000)[Chapter 20], Phillips and Hansen (1990) and Kitamura and Phillips (1997). I.e. (i) the number of instruments is larger or equal to the number of parameters, (ii) the  $T_\star$ -limit of  $\frac{1}{T_\star^2} \sum_{t=1}^{T_\star} (y_{it}^*, \tilde{\mathbf{x}}_{it}')' ((\tilde{\mathbf{x}}_{it}^*)', \tilde{\mathbf{x}}_{it}')$  is of rank  $k + 1$  (almost surely) and (iii) the  $T_\star$ -limit of  $\frac{1}{T_\star^2} \sum_{t=1}^{T_\star} ((\tilde{\mathbf{x}}_{it}^*)', \tilde{\mathbf{x}}_{it}')' ((\tilde{\mathbf{x}}_{it}^*)', \tilde{\mathbf{x}}_{it}')$  is of rank  $k + q_\rho$  (almost surely).

By our model assumptions it follows that  $\tilde{\mathbf{x}}_{it}^*$  has an impact on  $\tilde{y}_{it}^*$  (if  $\beta \neq 0$ ). The independence of  $\tilde{\mathbf{x}}_{it}^*$  and  $\tilde{\mathbf{u}}_{it}$  follows from the construction of  $\tilde{\mathbf{u}}_{it}$  (see Equation (5) and Assumption 2) and the fact that  $W_{ii} = 0$  (see Assumption 1). Appendix B shows that with  $\tau_v = 1$  and some regularity conditions on  $\mathbf{W}$  the conditions (ii) and (iii) on the rank are satisfied.<sup>5</sup>

To keep the notation simple, we consider - as already stated at the beginning of Section 2 - a model with  $k_\rho = 1$ . With  $q_\rho = k_\rho = 1$  we are in the just identified case, while if  $q_\rho > k_\rho = 1$  we consider the over-identified case. We collect our instruments in the  $nT_\star \times q_\rho$  matrix

$$\mathbf{x}^* = \left( \tilde{\mathbf{x}}_{11}^{*'}, \dots, \tilde{\mathbf{x}}_{1T_\star}^{*'}, \dots, \tilde{\mathbf{x}}_{n1}^{*'}, \dots, \tilde{\mathbf{x}}_{nT_\star}^{*'} \right)', \quad (19)$$

The set of our instruments is then  $\mathbf{Z} = (\mathbf{x}^*, \mathbf{x}, \zeta)$ . The matrix of explanatory variables  $\mathbf{X}$  is of dimension  $T_\star n \times 1 + k + (2p + 1)k \cdot n$ , while the dimension of  $\mathbf{Z}$  is  $T_\star n \times q_\rho + k + (2p + 1)k \cdot n$ .<sup>6</sup> Before we present our estimator, let us discuss why e.g. DOLS does not provide us with a consistent estimator:

**Remark 2.** [Endogeneity] Let us consider (16). From the discussion in the last paragraph of Remark 1 we already know that  $\tilde{u}_{it}^\dagger$  is correlated with  $\tilde{y}_{it}^*$  and with  $\tilde{\mathbf{x}}_{it}$ . Given the  $k + 1 + (2p + 1)k \cdot n$  dimensional

<sup>5</sup>In Phillips and Hansen (1990) and Kitamura and Phillips (1997) the requirements (ii) and (iii) in Assumption 5 are fulfilled by their construction of the model.

<sup>6</sup>A special case of our model is  $\Psi_i^\dagger(L) = \Psi^\dagger(L)$  for  $i = 1, \dots, n$ . Then  $\mathbf{X}$  is of dimension  $T_\star n \times 1 + k + (2p + 1)k$  while  $\mathbf{Z}$  is of dimension  $T_\star n \times q_\rho + k + (2p + 1)k$ .

vector of regressors  $\mathbf{X}_{it} = \left( \tilde{y}_{it}^*, \tilde{\mathbf{x}}'_{it}, \mathbf{0}_{1 \times (2p+1)k \cdot (i-1)}, \tilde{\delta}'_{it}, \mathbf{0}_{1 \times (2p+1)k \cdot (n-i-1)} \right)'$  and the  $k + q_\rho + (2p + 1)k \cdot n$  dimensional vector of instruments  $\mathbf{Z}_{it} = \left( \tilde{\mathbf{x}}^{*'}_{it}, \tilde{\mathbf{x}}'_{it}, \mathbf{0}_{1 \times (2p+1)k \cdot (i-1)}, \tilde{\delta}'_{it}, \mathbf{0}_{1 \times (2p+1)k \cdot (n-i-1)} \right)'$  we observe that  $\tilde{u}_{it}^\dagger$  is still correlated with  $\mathbf{X}_{it}$ . Therefore *DOLS* does not result in consistent estimates. When applying *2SLS* we have  $\mathbf{X}_{it}^{2SLS} = (\tilde{y}_{it}^*, \tilde{\mathbf{x}}'_{it})'$  and  $\mathbf{Z}_{it}^{2SLS} = (\tilde{\mathbf{x}}^{*'}_{it}, \tilde{\mathbf{x}}'_{it})'$ .  $\tilde{\mathbf{x}}'_{it}$  contained in  $\mathbf{Z}_{it}^{2SLS}$  is still correlated with  $\tilde{u}_{it}^\dagger$  by the model assumptions, such that also *2SLS* does not provide with a consistent estimator.

In analogy to a standard regression setting with endogenous regressors, we now construct a two stage-least square procedure for our panel setting. With two-stage least squares the initial stage results in projected values

$$\widehat{\mathbf{y}}^* = \mathbf{x}^* (\mathbf{x}^{*'} \mathbf{x}^*)^{-1} \mathbf{x}^{*'} \mathbf{y}^*, \quad (20)$$

while the second stage estimator is

$$\begin{pmatrix} \widehat{\rho} \\ \widehat{\beta} \\ \widehat{\delta} \end{pmatrix}_{D2SLS} = \begin{pmatrix} \widehat{\mathbf{y}}^{*'} \widehat{\mathbf{y}}^* & \mathbf{x}' \widehat{\mathbf{y}}^* & \zeta' \widehat{\mathbf{y}}^* \\ \widehat{\mathbf{y}}^{*'} \mathbf{x} & \mathbf{x}' \mathbf{x} & \zeta' \mathbf{x} \\ \widehat{\mathbf{y}}^{*'} \zeta & \mathbf{x}' \zeta & \zeta' \zeta \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\mathbf{y}}^{*'} \\ \mathbf{x}' \\ \zeta' \end{pmatrix} \mathbf{y}. \quad (21)$$

Since our instruments  $\tilde{\mathbf{x}}^*_{it}$  depend on  $u_{jt}^\dagger$  for  $j = 1, \dots, n$ , where  $j \neq i$  ( $W_{ii} = 0$  by Assumption 1) and  $u_{it}^\dagger$  and  $u_{jt}^\dagger$  are independent by Assumption 2, our instruments  $\tilde{\mathbf{x}}^*_{it}$  are orthogonal to  $\tilde{\mathbf{u}}_{it}$  and  $\tilde{u}_{it}$ . In the first stage we project the endogenous variable  $\tilde{y}_{it}^*$  on  $\tilde{\mathbf{x}}^*_{it}$ . The projected values, denoted by  $\widehat{\tilde{y}}_{it}^*$ , will also be orthogonal to  $\tilde{\mathbf{u}}_{it}$  and  $\tilde{u}_{it}$ .  $\tilde{\mathbf{x}}_{it}$  and  $\tilde{\mathbf{u}}_{it}$  are asymptotically orthogonal by Assumption 4. Our analysis will show that this results in a consistent estimator in the second stage regression. Based on this discussion, we can now compactly write the *dynamic two-stage least squares* estimator of  $(\rho, \beta', \delta')' = (\gamma', \delta')'$  as

$$\begin{aligned} (\widehat{\gamma}', \widehat{\delta}')'_{D2SLS} &= (\mathbf{X}' P_H \mathbf{X})^{-1} \mathbf{X}' P_H \mathbf{y} \\ &= (\gamma', \delta')' + (\mathbf{X}' P_H \mathbf{X})^{-1} \mathbf{X}' P_H \mathbf{u}. \end{aligned} \quad (22)$$

$P_H$  is the project operator (see e.g. Ruud (2000)[Chapter 3]), projecting on the space spanned by  $\mathbf{Z}$ , i.e.

$$P_H = \mathbf{Z} (\mathbf{Z}' \mathbf{Z})^{-1} \mathbf{Z}'. \quad (23)$$

Since  $\mathbf{Z}$  is a  $T_\star n \times q_\rho + k + (2p + 1)k \cdot n$  matrix,  $P_H$  has to be a  $T_\star n \times T_\star n$  matrix. With  $q_\rho = 1$ , we are in the just identified case, where the estimator is given by:

$$\widehat{(\gamma', \delta')}_{D2SLS}' = (\mathbf{Z}'\mathbf{X})^{-1} \mathbf{Z}'\mathbf{y} = (\gamma', \delta')' + (\mathbf{Z}'\mathbf{X})^{-1} \mathbf{Z}'\mathbf{u}. \quad (24)$$

Equipped with the  $D2SLS$  estimator (22) we obtain the following result:

**Theorem 1** (Limits for  $D2SLS$  Estimation). *Consider the fixed effects spatial correlation model (1) and the estimator (22) based on the within-transformed model (9). Suppose that the Assumptions 1 to 5 hold.  $T_\star = T - 2p(T)$ . Then for  $n$  fixed and  $T \rightarrow \infty$  it follows that*

1.  $T_\star(\hat{\gamma}_{D2SLS} - \gamma)$  and  $\sqrt{T_\star}(\hat{\delta}_{D2SLS,i} - \delta_i)$  are asymptotically independent for each  $i = 1, \dots, n$ .
2.  $\sqrt{n}T_\star(\hat{\gamma}_{D2SLS} - \gamma)$  converges in distribution to  $M_n^{-1}m_n$  where  $m_n$  and  $M_n$  are given by (58) and (59).
3. Given a  $s \times k + 1$  restriction matrix  $\mathbf{R}$ , the Wald statistic  $S_{\gamma,nT}$  defined in (65) converges to a  $\chi^2$  random variable with  $s$  degrees of freedom.

**Remark 3.** By Assumption 4, if  $T \rightarrow \infty$ , then  $T_\star \rightarrow \infty$ . In addition, the reader should note that the two-stage least squares estimator and the  $DOLS$  estimator are special cases of the dynamic two-stage least squares estimator. With an empty matrix  $\zeta$  we get the two-stage least squares estimator.<sup>7</sup> If  $\mathbf{x}^* = \mathbf{y}^*$ , then we obtain the  $DOLS$  estimator. The Wald-statistic presented in Appendix A can be used to obtain the Wald statistic for the two-stage least squares estimator and the  $DOLS$  estimator.

## 4 Monte Carlo Simulations

This section investigates the small sample properties of the  $D2SLS$  estimator as well as the size and power of the Wald tests defined in Theorem 1. We generate the data based on an error process that follows from Assumptions 2-4. To operationalize this we need to specify the lag polynomials  $\Psi_i^\dagger(L)$ . In particular, we have to specify the error dynamics of the vector  $\mathbf{w}_{it}^\dagger$ . Here we assume the same error dynamics for all

<sup>7</sup>Note that this does not correspond to the  $p = 0$  case, where still  $\Delta\tilde{\mathbf{x}}_{it}$  is included in  $\zeta$ .

cross sections  $i = 1, \dots, n$ . We use two explanatory variables  $x_{it}$  such that  $k = 2$  and set  $\beta = (1, 1)'$ . The number of instruments is  $k_\rho = 2$ .

Regarding the error dynamics we use the stationary designs of Binder et al. (2005) to generate the data for the vector  $\mathbf{w}_{it}^\dagger$ . The innovations  $\varepsilon_{it}^\dagger$  are generated as independent draws from  $\varepsilon_{it}^\dagger \sim N(0, \Sigma_{\varepsilon i})$ . For  $\Sigma_{\varepsilon i}$  we use (I)  $[\Sigma_{\varepsilon i}]_{jj} = 1$  for  $j = 1, \dots, 3$ , the remaining elements are  $-0.2$ , (II)  $\Sigma_{\varepsilon i} = \mathbf{I}_3$  and (III)  $[\Sigma_{\varepsilon i}]_{jj} = 1$  for  $j = 1, \dots, 3$ , while the other elements are  $0.2$ . In a first three designs we generate  $\mathbf{w}_{it}^\dagger$  by means of the first order vector autoregressive system ( $VAR(1)$ )

$$\mathbf{w}_{it}^\dagger = \Phi \mathbf{w}_{i,t-1}^\dagger + \varepsilon_{it}^\dagger. \quad (25)$$

For the  $3 \times 3$  matrix  $\Phi$  we use the following designs:

Design  $DGP = 1$ : A stationary  $VAR(1)$  with maximum eigenvalue of 0.6

$$\Phi = \begin{pmatrix} 0.4 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0.1 \\ 0.1 & 0.1 & 0.4 \end{pmatrix}. \quad (26)$$

Design  $DGP = 2$ : A stationary  $VAR(1)$  with maximum eigenvalue of 0.8

$$\Phi = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}. \quad (27)$$

Design  $DGP = 3$ : A stationary  $VAR(1)$  with maximum eigenvalue of 0.95

$$\Phi = \begin{pmatrix} 0.75 & 0.1 & 0.1 \\ 0.1 & 0.75 & 0.1 \\ 0.1 & 0.1 & 0.75 \end{pmatrix}. \quad (28)$$

In addition we consider a finite-order vector moving average processes of the form

$$\mathbf{w}_{it}^\dagger = \varepsilon_{it}^\dagger + \sum_{l=1}^q \Psi_{il}^\dagger \varepsilon_{i,t-l}^\dagger, \quad (29)$$

where we choose: Design  $DGP = 4$ , which is a first-order MA process where

$$\Psi_{i1}^\dagger = \begin{pmatrix} 0.4 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0.1 \\ 0.1 & 0.1 & 0.4 \end{pmatrix}, \quad (30)$$

and Design  $DGP = 5$ , where  $\mathbf{w}_{it}^\dagger$  follows a second-order MA process with

$$\Psi_{i1}^\dagger = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.1 & 0.6 \end{pmatrix} \text{ and } \Psi_{i2}^\dagger = \begin{pmatrix} 0.4 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0.1 \\ 0.1 & 0.1 & 0.4 \end{pmatrix}. \quad (31)$$

Recall that the disturbance of  $y_{it}$  is given by the first element of the vector  $\mathbf{w}_{it}^\dagger$ , while its remaining elements result in  $\mathbf{x}_{it}$ . The maximum numbers of leads and lags of the explanatory variables that are conditionally correlated with the disturbances is equal to one in the Designs 1-3, while for the Designs 4 and 5 they remain different from zero.

In the case of the VAR(1) models, we generate the initial values for the process  $\mathbf{w}_{it}^\dagger$  from the implied stationary distribution. Note that by backward substitution, we obtain

$$\mathbf{w}_{i0}^\dagger = \sum_{j=0}^{\infty} \Phi^j \varepsilon_{i,-j}^\dagger. \quad (32)$$

and hence  $\mathbf{w}_{i0}^\dagger$  is a random variable that is independent from  $\varepsilon_{it}^\dagger$  for  $t > 0$ . When the innovations  $\varepsilon_{it}^\dagger$  are normally distributed, it also follows that  $\mathbf{w}_{i0}^\dagger$  is normally distributed. Furthermore, it has a mean of zero

and  $k + 1 \times k + 1$  variance-covariance matrix  $E(\mathbf{w}_{it}^\dagger \mathbf{w}_{it}^{\dagger'}) = \mathbf{\Gamma}_{i0}^\dagger$  where

$$\mathbf{\Gamma}_{i0}^\dagger = \mathbb{E} \left[ \left( \sum_{j=0}^{\infty} \mathbf{\Phi}^j \varepsilon_{i,-j}^\dagger \right) \left( \sum_{j=0}^{\infty} \mathbf{\Phi}^j \varepsilon_{i,-j}^\dagger \right)' = \sum_{j=0}^{\infty} \mathbf{\Phi}^j \Sigma_{\varepsilon i} \mathbf{\Phi}'^j \right]. \quad (33)$$

The above expression implies

$$\mathbf{\Phi} \mathbf{\Gamma}_{i0}^\dagger \mathbf{\Phi}' = \sum_{j=0}^{\infty} \mathbf{\Phi}^{j+1} \Sigma_{\varepsilon i} \mathbf{\Phi}'^{j+1} = \Sigma_{\varepsilon i} + \sum_{j=0}^{\infty} \mathbf{\Phi}^j \Sigma_{\varepsilon i} \mathbf{\Phi}'^j = \Sigma_{\varepsilon i} + \mathbf{\Gamma}_{i0}^\dagger. \quad (34)$$

After vectorizing and solving for  $\mathbf{\Gamma}_{i0}^\dagger$  we obtain (see e.g. Lütkepohl (2006)[p. 29])

$$\text{vec}(\mathbf{\Gamma}_{i0}^\dagger) = (\mathbf{I}_{(k+1)^2} - \mathbf{\Phi} \otimes \mathbf{\Phi})^{-1} \text{vec}(\Sigma_{\varepsilon i}). \quad (35)$$

The remaining parameters of the model are chosen as follows: We generate the individual effects  $\alpha_i$  from  $\alpha_i \sim N(\mathbf{0}_{3 \times 1}, \mathbf{I}_3)$ . The spatial correlation parameter  $\rho$  is chosen from the set  $\{-0.95, -0.5, -0.1, 0, 0.1, 0.5, 0.95\}$ . The choice of  $\mathbf{W}$  is based on Kapoor et al. (2007). In more details we consider: (i) A "one step ahead-one step behind circular world" with corresponding entries 1/2. I.e.  $W_{i,i+1} = 0.5$  and  $W_{i+1,i} = 0.5$  for  $i = 1, \dots, n-1$ .  $W_{1,n} = 0.5$  and  $W_{n,1} = 0.5$ , the other entries are zero. (ii) A "three step ahead-three step behind circular world" with corresponding entries 1/6. (iii) A "five step ahead-five step behind circular world" with corresponding entries 1/10. (iv) A "one step ahead-one step behind Rook constellation" with corresponding entries 1/2. This design is non-circular. Here  $W_{i,i+1} = 0.5$  and  $W_{i+1,i} = 0.5$  for  $i = 1, \dots, n-1$ ; the other entries are zero. (v) A "two step ahead-two step behind Queen constellation". In this non-circular design  $W_{i,i+1} = 0.3$ ,  $W_{i,i+2} = 0.2$ ,  $W_{i+1,i} = 0.3$  and  $W_{i+2,i} = 0.2$  for  $i = 1, \dots, n-2$ ; the other entries are zero. Thus we have in total 525 different data generating processes (3, 5, 7, 5 different settings for  $\Sigma_{\varepsilon i}$ , the autoregressive structure of  $\mathbf{w}_{it}^\dagger$ , the spatial correlation parameter  $\rho$  and the spatial correlation matrix  $\mathbf{W}$ , respectively).

For the estimation of the long run covariance  $\Omega_{uu,i}$  we applied the Bartlett and the truncated kernel (given these kernels and our model assumptions the conditions for consistent estimation provided in



Jansson (2002) are satisfied). I.e.  $\hat{\Omega}_{uu,i} = \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T k\left(\frac{t-s}{b_T}\right) \hat{u}_{it}^2$ , where  $k(\cdot)$  is a kernel function with bandwidth  $b_T$  and  $\hat{u}_{it}$  are the residuals. The truncated kernel exhibits a better performance than the Bartlett kernel. For the truncated kernel we either kept the number of lags fixed for all  $i = 1, \dots, n$  or stopped the summation for component  $i$  at lag  $s$  if the mode of the autocorrelation of the residuals  $\hat{u}_{it}, \hat{u}_{i,t-s}$  becomes smaller than the  $1.96/\sqrt{T}$  bound. The better performance has been observed with the second opportunity. When implementing dynamic least squares also the leads and lags  $p$  have to be chosen. Recent literature proposed to choose  $p$  by means of information criteria (see e.g. Kejriwal and Perron (2008) and Kurozumi and Tuvaandorj (2010)). With small  $T$  and  $n$  the implementation of such criteria is not difficult. However, since a data set with (relatively) large  $n$  and  $T$  is going to be considered, working with small  $p$  becomes necessary due to computational restrictions. In more detail we set  $p = 2$  for all components  $i = 1, \dots, n$ . For all designs working with  $p = 2$  performed better than working with  $p = 1$ . Last but not least  $M = 1000$  is the number of Monte Carlo steps and  $m$  is the index of the corresponding step. To obtain the *DOLS* estimates  $\mathbf{Z}$  is replaced by  $\mathbf{X}$  in (24). For the *2SLS* estimator  $\mathbf{Z}$  and  $\mathbf{X}$  do not contain any leads and lags, i.e.  $\zeta = \emptyset$ , while for *OLS* we use  $\mathbf{Z} = \mathbf{X}$  with  $\zeta = \emptyset$ .

Tables 1 to 8 present results from the simulation runs. We consider the cases where  $n = 5, T = 200$  and  $n = 50, T = 200$ . In the first columns the numbers abbreviate the different designs:  $\mathbf{W} = (i)$  for the one-steps ahead setting, ...,  $\mathbf{W} = (v)$  for the Queen constellation,  $DGP = 1, \dots, 5$  for the autoregressive model used to generate  $\mathbf{w}_{it}$ ,  $\Sigma_\varepsilon = (I), (II), (III)$  stands for the covariance matrix used. With different cross-sectional dimensions  $n$  we investigated the size of the Wald statistic. I.e. we obtained the percentages of the simulation runs, where the true null hypothesis  $\rho = 0$  has been rejected by applying  $\alpha_c = \{0.01, 0.05, 0.1\}$  significance levels. For  $\rho = \{-0.95, -0.5, -0.1, 0.1, 0.5, 0.95\}$  the *false* null-hypothesis of  $\rho = 0$  has been rejected in all of the simulation runs for most simulation settings. The exceptions appear with  $n = 5$ ,  $\rho = \pm 0.1$  and the moving average designs  $DGP = 4$  and  $DGP = 5$ . Here for different  $\mathbf{W}$  and  $\Sigma_\varepsilon$  also rejection rates smaller than 99% are observed. The smallest rejection rate was 0.45, which was observed for the *MA(1)* process, where  $\mathbf{W}$  of type (iii) and  $\Sigma_\varepsilon$  of type (I). For  $n = 50$  we observed that the false null hypothesis  $\rho = 0$  was never rejected.

Tables 1, 2, 5 and 5 present the rejection rates of the Wald test for the true null hypothesis  $\rho = 0$ . Since

for  $\rho = 0$  the *DOLS* estimation procedure is one of the theoretically correct tools presented in literature, the comparison of *D2SLS* to *DOLS* is of special interest. With  $n = 5$ , for designs  $DGP = 1 - 3$  the oversizing remains modest for *DOLS* and *D2SLS*. The rejection rates observed are very similar although the *D2SLS* uses the instrumental variables, where numerical complexity is increased. With the moving average process stronger oversizing effects in the size are obtained, for *DOLS* these effect can become very strong. The highest oversizing with our *D2SLS* estimator is observed with  $\Sigma_\varepsilon = (I)$ , where the "serial" endogeneity is stronger and  $\mathbf{W} = (iii)$ , where many spatial lags are used. For  $n = 50$  the performance of *DOLS* is very close to the performance of our *D2SLS* estimator; in some settings undersizing is observed.

According to the  $Bias = \rho - \frac{1}{M} \sum_{m=1}^M \hat{\rho}_m$  and the bias adjusted root mean squared error  $RMSE = \sqrt{Bias^2 + \left( \frac{1}{M-1} \sum_{m=1}^M \hat{\rho}_m^2 - \left( \frac{1}{M} \sum_{m=1}^M \hat{\rho}_m \right)^2 \right)}$  hardly any differences are observed between *DOLS* and *D2SLS* with  $n = 5$  for the designs  $DGP = 1 - 3$ , while substantial differences can be observed with  $DGP = 4$  and 5 (see Tables 3 and 4 for  $\rho = 0.5$ ). In addition these effects are more pronounced with  $\Sigma_\varepsilon = (I)$  and  $\mathbf{W} = (iii)$ . Surprisingly, *2SLS* - sometimes also *OLS* - worked quite well for the different designs considered although, these methods do not cope with "serial" endogeneity still arising with  $\rho = 0$ . We explain this phenomenon as follows: The Wald statistic  $\mathcal{W}$  includes estimates of the long-run covariance matrix  $\Omega_{uu,i}$  and the parameter estimates. We observed that the estimates of the covariance matrix with *OLS* and *2SLS* are in almost all cases much larger than the estimates with *DOLS* and *D2SLS*. However, also the volatility of the estimator, e.g. measured by the RMSE, increases. Here we claim that these effects can cancel out, such that the rejection rates are not too far away from the significance level  $\alpha_c$  considered. With *D2SLS* the rejection rates are in a lot of the cases even closer to  $\alpha_c$  than with *DOLS* and *D2SLS*. Oversizing of Wald tests based on *DOLS* estimation (which also require the calculation of the long run covariance matrix) have already been extensively reported in literature (e.g. Mark and Sul (2003)[Table 1]). For  $n = 50$  the bias and the *RMSE* become smaller, as can expected when using more data (see Tables 7 and 8 for  $\rho = 0.95$ ). Although most differences are small, we observe that the bias and the *RMSE* of our *D2SLS* estimator is smaller or equal to the bias and the *RMSE* of the *DOLS* estimator. In addition - except for  $W = (v)$ ,  $DPG = 4$  and  $\Sigma_\varepsilon = (I)$ , *D2SLS* dominates the *2SLS* estimator. *D2SLS* always dominates *OLS* in terms of the bias and the *RMSE*.

Since *2SLS* did not perform so bad in a lot of the cases considered above, we also increased the degree of serial "endogeneity". We additionally investigate (*IV*) a lower triangular Cholesky decomposition of  $\Sigma_{\varepsilon i}$  where  $[Chol(\Sigma_{\varepsilon i})]_{jj} = 1$  for  $j = 1, \dots, 3$ , the elements below the main diagonal are 0.9, and (*V*) a symmetric  $\Sigma_{\varepsilon i}$  where  $[\Sigma_{\varepsilon i}]_{jj} = 1$  for  $j = 1, \dots, 3$  where the other elements are 0.9. For  $n = 5$ , even with  $\rho = 0$  the bias with *2SLS* is a factor two to ten larger than the bias observed with *D2SLS*. For  $\rho \neq 0$  the bias increases. Things are even worse with *OLS*. Regarding our *D2SLS* estimator the usual oversizing behavior with  $\rho = 0$  is also observed here. Regarding the empirical analysis carried out in the next section, it is important to note that even with  $\rho = 0$  the bias with *2SLS* for the estimates of the regressors  $\beta$  is approximately a factor two the three of the bias observed with *D2SLS*. This can be explained by the fact that  $\tilde{\mathbf{x}}_{it}$  is part of  $\mathbf{Z}_{it}$ , where  $\tilde{\mathbf{x}}_{it}$  is correlated with the noise term by the construction of the model. For  $n = 50$  the bias in most cases is the smallest one with *D2SLS*, however the differences in absolute terms become small with  $n = 50$ .

Summing up, we observe that the estimator developed in Section 3 exhibits (in most cases) some oversizing behavior as already observed with a lot of literature where dynamic least squares estimation has been applied. However, even with the true null-hypothesis  $\rho = 0$ , where no spatial endogeneity is present, the *D2SLS* estimator in most cases performs better than the *DOLS* estimator. By increasing, both - serial and spatial endogeneity - we observe that the *D2SLS* estimator performs reasonably well. With the larger data sets, where  $T = 200$  and  $n = 50$ , the bias and the *RMSE* decreases compared to  $T = 200$  and  $n = 5$ .

## 5 Empirical Illustration

In this section we apply the tools developed in the former sections to credit risk data. Quantitative finance literature has mainly focused on the default risk of the entity (see e.g. Eom et al. (2004), Crosbie and Bohn (2003), Collin-Dufresne et al. (2001), Campbell and Taksler (2003), Ericsson et al. (2009), Longstaff et al. (2008), among others). In their seminal paper Collin-Dufresne et al. (2001) looked at the residuals – arising from regressing bond spreads on usual credit risk factors – by means of a principal component analysis, where the authors detected a strong factor in the residuals. While the coefficients

<i>W</i>	DGP	$\Sigma_\varepsilon$	$\alpha_c = 0.01$				$\alpha_c = 0.05$				$\alpha_c = 0.10$			
			<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>	<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>	<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>
i	1	I	1.0	1.0	3.0	2.9	5.4	5.7	9.2	9.4	10.8	11.0	15.1	15.2
i	1	II	1.2	1.2	2.4	2.2	5.3	5.3	8.7	8.8	10.9	11.2	15.1	14.9
i	1	III	1.2	1.2	3.6	3.5	6.0	5.8	8.9	9.0	11.1	11.2	14.9	14.7
i	2	I	1.7	1.6	2.5	2.5	4.7	4.7	7.8	7.7	8.3	8.4	12.0	12.1
i	2	II	1.8	1.8	3.4	3.2	5.5	5.6	10.1	10.3	9.0	9.1	16.8	16.5
i	2	III	2.6	2.6	3.8	3.8	6.3	6.3	11.8	12.0	11.7	11.7	18.1	17.5
i	3	I	1.6	1.6	1.9	2.0	4.4	4.5	6.8	6.7	8.3	8.4	10.9	11.0
i	3	II	1.8	1.8	2.3	2.3	5.7	5.6	9.2	9.0	8.8	8.9	15.2	15.2
i	3	III	3.0	2.9	3.6	3.6	6.2	6.4	11.4	11.4	11.5	11.4	17.7	17.6
i	4	I	1.4	1.4	1.6	1.6	4.8	5.0	6.6	6.8	9.7	8.1	12.8	12.9
i	4	II	1.2	1.3	2.0	2.1	4.5	4.7	7.1	6.9	8.7	8.7	12.5	12.5
i	4	III	1.3	1.3	2.0	2.0	4.9	4.8	7.0	7.2	10.4	10.4	12.6	12.6
i	5	I	3.8	3.1	7.2	7.8	10.5	9.2	17.2	16.7	18.6	15.4	23.5	23.8
i	5	II	3.1	3.0	5.0	5.4	9.6	9.5	13.8	13.5	16.8	16.8	21.7	21.6
i	5	III	3.2	3.2	5.0	5.0	10.2	10.4	12.5	12.5	16.5	16.6	21.4	21.4
ii	1	I	0.9	0.9	3.1	2.9	5.5	5.2	9.9	9.6	10.7	10.6	16.3	16.1
ii	1	II	1.4	1.3	3.0	2.6	6.3	6.0	9.7	9.6	12.3	12.5	17.4	17.3
ii	1	III	2.0	1.9	2.7	3.1	6.9	6.9	10.2	10.3	13.0	13.4	17.3	17.3
ii	2	I	1.1	1.1	3.2	3.0	3.3	3.2	8.2	8.0	6.8	6.8	14.7	14.4
ii	2	II	1.0	1.0	3.1	3.0	4.3	4.4	9.2	9.0	7.2	7.1	16.6	16.1
ii	2	III	1.5	1.6	3.6	3.7	5.7	5.7	10.0	9.6	10.3	10.4	16.2	16.4
ii	3	I	1.0	1.0	2.5	2.6	3.5	3.4	7.4	7.4	6.9	6.9	13.4	13.7
ii	3	II	1.0	1.0	2.6	2.6	4.3	4.1	8.8	8.9	6.7	6.9	15.2	14.8
ii	3	III	1.7	1.7	3.1	3.1	5.5	5.4	9.1	9.2	9.3	9.4	16.0	16.2
ii	4	I	1.3	1.2	2.0	1.8	4.8	3.9	7.2	7.5	10.4	8.8	12.6	11.9
ii	4	II	1.0	0.8	1.7	2.0	4.1	4.0	7.1	7.3	9.1	9.5	12.7	13.0
ii	4	III	1.4	1.3	2.2	2.2	4.7	4.9	7.5	7.4	10.6	10.6	12.3	12.6
ii	5	I	4.2	2.7	7.2	7.1	11.9	9.3	17.4	17.9	18.9	16.5	25.8	27.6
ii	5	II	3.2	3.2	5.5	5.2	10.3	10.6	13.9	14.2	17.4	17.3	21.3	21.8
ii	5	III	3.2	3.2	5.7	5.5	10.5	10.6	14.0	14.0	18.2	18.2	20.4	20.9
iii	1	I	2.1	0.9	4.6	2.9	8.3	5.2	13.0	9.7	15.0	10.7	19.6	16.1
iii	1	II	2.3	1.3	4.9	2.7	9.3	6.0	13.4	9.7	16.2	12.5	19.8	17.4
iii	1	III	3.0	1.8	5.4	3.1	11.0	6.9	15.9	10.5	18.3	13.4	22.7	17.3
iii	2	I	1.2	1.1	3.1	3.0	3.7	3.3	9.1	8.0	7.1	6.7	15.2	14.4
iii	2	II	1.1	1.0	3.9	3.0	4.3	4.4	10.9	8.9	7.8	7.4	17.7	16.2
iii	2	III	1.6	1.6	4.5	3.7	6.7	5.8	11.2	9.5	11.5	10.4	18.8	16.3
iii	3	I	1.0	1.0	2.6	2.6	3.8	3.4	8.5	7.4	6.7	6.9	13.7	13.6
iii	3	II	1.1	1.0	3.3	2.6	4.6	4.1	10.2	8.8	7.4	6.8	16.9	14.9
iii	3	III	2.0	1.7	4.1	3.1	6.2	5.4	10.7	9.2	10.5	9.3	17.8	16.4
iii	4	I	68.5	1.0	47.3	1.8	78.9	4.0	61.1	7.0	84.0	8.8	67.6	11.7
iii	4	II	5.0	0.9	6.0	2.0	11.4	4.0	14.4	7.3	18.7	9.3	21.8	13.0
iii	4	III	1.9	1.3	2.4	2.2	7.0	4.9	7.9	7.4	13.2	10.6	14.1	12.6
iii	5	I	75.5	2.4	61.0	7.1	84.9	9.1	69.9	17.7	88.7	16.2	76.1	27.5
iii	5	II	9.2	3.2	12.3	5.3	21.2	10.6	22.1	14.3	28.2	17.3	31.7	21.9
iii	5	III	4.6	3.2	6.2	5.5	13.4	10.6	14.4	14.0	22.3	18.2	22.5	20.9

**Table 1:** Size for the parameter  $\rho$ : Rejections rates of the Wald test for the true null hypothesis  $\rho = 0$  in percentage terms, given the significance levels  $\alpha_c = \{0.01, 0.05, 0.1\}$ .  $M = 1000$  simulation runs. Cross-sectional dimension  $n = 5$ , time series dimension  $T = 200$ .

of determination in the initial regressions are surprisingly low, this factor has a higher explanatory power than the regressors obtained from economic literature. Collin-Dufresne et al. (2001) claim that the strong factor is driven by liquidity risk or other joint market behavior. Based on these findings a lot of articles

$W$	Des.	$\Sigma_\varepsilon$	$\alpha_c = 0.01$				$\alpha_c = 0.05$				$\alpha_c = 0.10$			
			<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>	<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>	<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>
iv	1	I	0.9	0.9	3.1	3.1	4.6	4.7	8.6	8.2	10.0	9.9	15.5	15.7
iv	1	II	1.4	1.4	2.7	2.5	4.6	4.6	7.9	7.9	10.5	10.4	14.7	14.8
iv	1	III	1.3	1.3	3.0	2.9	5.0	5.2	9.0	9.6	11.0	11.1	15.1	15.1
iv	2	I	1.8	1.8	1.9	1.8	5.7	5.9	7.4	7.5	9.1	8.9	13.6	13.3
iv	2	II	2.3	2.1	3.2	3.1	6.6	6.6	10.6	10.6	10.2	10.2	17.9	18.2
iv	2	III	2.6	2.6	5.1	4.9	7.4	7.6	13.0	12.9	12.5	12.3	18.8	19.0
iv	3	I	1.7	1.7	1.7	1.7	5.4	5.3	7.0	7.2	8.7	9.0	11.7	11.6
iv	3	II	2.2	2.1	2.8	2.8	6.3	6.5	9.5	9.6	9.8	9.9	16.4	16.6
iv	3	III	2.4	2.4	4.1	4.5	7.0	6.9	11.6	11.8	11.4	11.2	17.7	17.8
iv	4	I	1.3	1.5	2.1	2.3	4.8	5.0	7.7	7.0	10.9	9.3	13.7	12.4
iv	4	II	1.0	1.0	2.2	2.5	5.2	5.5	7.1	7.5	10.6	10.4	13.8	14.4
iv	4	III	1.5	1.5	2.3	2.3	6.1	6.2	8.4	8.5	11.3	11.2	13.9	14.2
iv	5	I	4.5	3.9	7.0	7.7	11.4	10.1	17.0	16.9	19.2	16.6	23.8	23.7
iv	5	II	4.1	4.0	6.1	6.1	10.8	10.7	13.8	13.8	17.3	17.2	21.2	20.9
iv	5	III	4.2	4.2	6.0	5.9	11.2	11.4	13.9	14.2	18.1	18.2	21.0	20.9
v	1	I	0.9	0.9	3.2	3.3	5.1	5.0	9.4	9.4	9.9	10.0	15.8	15.9
v	1	II	1.1	1.1	2.6	2.6	5.5	5.6	8.9	8.9	10.8	10.7	14.5	14.4
v	1	III	1.4	1.4	2.7	2.7	6.6	6.8	9.9	10.1	12.3	12.3	17.4	17.3
v	2	I	1.4	1.4	2.5	2.5	5.1	5.4	7.7	7.9	9.6	9.6	13.2	12.8
v	2	II	1.4	1.4	3.1	3.1	6.1	6.1	11.2	11.3	9.7	9.4	17.6	18.0
v	2	III	2.3	2.3	3.9	3.8	7.3	7.2	12.6	12.6	13.0	12.9	19.2	19.2
v	3	I	1.0	1.0	2.0	2.0	5.1	5.1	6.6	6.6	9.6	9.4	11.9	11.8
v	3	II	1.4	1.4	2.7	2.7	5.9	5.6	9.4	9.5	9.8	9.8	15.9	16.2
v	3	III	2.1	2.1	3.4	3.3	6.8	6.7	12.0	12.2	12.9	12.7	18.7	18.8
v	4	I	1.7	1.0	2.7	2.4	5.2	3.9	7.6	7.0	10.0	8.1	13.6	12.6
v	4	II	0.7	0.7	2.2	2.1	4.9	4.6	7.1	7.0	9.7	9.8	12.7	13.4
v	4	III	0.4	0.4	2.0	2.0	5.2	5.1	7.4	7.3	11.3	11.4	13.1	13.2
v	5	I	3.9	2.9	7.6	7.8	9.9	9.2	16.9	17.1	17.2	16.9	25.7	24.9
v	5	II	2.8	3.0	5.7	5.9	11.0	10.7	14.3	13.8	17.8	17.6	22.3	22.4
v	5	III	3.0	2.9	6.0	6.0	11.8	11.8	14.1	14.1	18.9	18.7	22.0	22.2

**Table 2:** Size for the parameter  $\rho$ : Rejections rates of the Wald test for the true null hypothesis  $\rho = 0$  in percentage terms, given the significance levels  $\alpha_c = \{0.01, 0.05, 0.1\}$ .  $M = 1000$  simulation runs. Cross-sectional dimension  $n = 5$ , time series dimension  $T = 200$ .

also looked on joint determinants of credit spreads (see e.g. Zhou (2001), Collin-Dufresne et al. (2003), Jorion and Zhang (2007) and Norden and Weber (2009)). In the following a spatial correlation matrix  $\mathbf{W}$  will be derived from input-output data. Equipped with this matrix  $\mathbf{W}$  we shall estimate model (9) by the *D2SLS* approach. Similar to Berndt et al. (2008) the left hand side variable is the CDS spread, while firm specific credit risk proxies and the *VIX* volatility index are used as the right hand side variables. By means of the matrix  $\mathbf{W}$  we model some form of default risk correlation. The Wald test developed in Theorem 1 checks whether the impact of spatial correlation described by  $\mathbf{W}$  is significant. Although our approach cannot "solve" the economic problem highlighted by Collin-Dufresne et al. (2001), the following analysis tries to add a further part to the puzzle of modeling credit spreads.

W	DGP	$\Sigma_\epsilon$	Bias				RMSE			
			OLS	2SLS	DOLS	D2SLS	OLS	2SLS	DOLS	D2SLS
i	1	I	-5.3E-4	2.4E-5	-4.3E-4	1.6E-5	2.16E-3	2.04E-3	2.04E-3	1.95E-3
i	1	II	-5.9E-4	1.0E-5	-5.7E-4	2.0E-5	2.32E-3	2.15E-3	2.15E-3	2.36E-3
i	1	III	-5.9E-4	1.1E-4	-6.0E-4	1.4E-4	2.44E-3	2.37E-3	2.37E-3	2.61E-3
i	2	I	-3.7E-4	-9.7E-5	-2.0E-4	-6.2E-5	2.44E-3	2.60E-3	2.60E-3	1.25E-3
i	2	II	-4.3E-4	-1.5E-4	-2.9E-4	-1.0E-4	2.49E-3	2.63E-3	2.63E-3	1.55E-3
i	2	III	-4.4E-4	-1.1E-4	-4.2E-4	-1.3E-4	2.66E-3	2.80E-3	2.80E-3	2.11E-3
i	3	I	-3.6E-4	-9.0E-5	-1.7E-4	-4.8E-5	2.38E-3	2.55E-3	2.55E-3	1.04E-3
i	3	II	-4.2E-4	-1.4E-4	-2.4E-4	-7.7E-5	2.40E-3	2.54E-3	2.54E-3	1.27E-3
i	3	III	-4.2E-4	-9.6E-5	-3.5E-4	-1.1E-4	2.50E-3	2.64E-3	2.64E-3	1.71E-3
i	4	I	-4.2E-2	-8.0E-4	-2.0E-2	-6.2E-4	5.93E-2	1.86E-2	1.86E-2	2.95E-2
i	4	II	-8.4E-4	-7.2E-5	-8.4E-4	-6.9E-5	2.50E-3	2.10E-3	2.10E-3	2.57E-3
i	4	III	-1.4E-4	-1.3E-5	-1.0E-4	-1.9E-5	7.18E-4	7.15E-4	7.15E-4	6.81E-4
i	5	I	-4.2E-2	-8.5E-4	-2.0E-2	-7.0E-4	5.97E-2	1.84E-2	1.84E-2	2.96E-2
i	5	II	-8.5E-4	-7.8E-5	-8.4E-4	-8.3E-5	2.50E-3	2.10E-3	2.10E-3	2.57E-3
i	5	III	-1.4E-4	-1.5E-5	-9.7E-5	-2.7E-5	7.17E-4	7.10E-4	7.10E-4	6.89E-4
ii	1	I	-9.7E-4	-2.9E-4	-5.7E-4	3.4E-5	4.98E-3	4.66E-3	4.66E-3	4.44E-3
ii	1	II	-8.9E-4	-9.6E-5	-8.1E-4	-1.3E-5	5.44E-3	5.21E-3	5.21E-3	5.73E-3
ii	1	III	-8.1E-4	3.0E-4	-1.2E-3	-1.2E-4	6.11E-3	6.01E-3	6.01E-3	6.83E-3
ii	2	I	-7.1E-4	-4.7E-4	-3.1E-4	-1.5E-4	5.12E-3	5.06E-3	5.06E-3	2.96E-3
ii	2	II	-7.9E-4	-5.1E-4	-4.9E-4	-2.7E-4	5.26E-3	5.17E-3	5.17E-3	3.60E-3
ii	2	III	-6.7E-4	-3.0E-4	-7.0E-4	-3.7E-4	5.68E-3	5.63E-3	5.63E-3	4.69E-3
ii	3	I	-7.0E-4	-4.7E-4	-2.5E-4	-1.1E-4	4.96E-3	4.90E-3	4.90E-3	2.45E-3
ii	3	II	-7.7E-4	-5.0E-4	-4.0E-4	-2.0E-4	5.03E-3	4.95E-3	4.95E-3	2.95E-3
ii	3	III	-6.6E-4	-3.2E-4	-5.7E-4	-2.9E-4	5.28E-3	5.22E-3	5.22E-3	3.79E-3
ii	4	I	-5.2E-2	7.0E-3	-2.5E-2	-1.5E-4	7.88E-2	5.40E-2	5.40E-2	4.48E-2
ii	4	II	-1.7E-3	-3.9E-4	-1.5E-3	-1.5E-4	6.53E-3	5.81E-3	5.81E-3	6.60E-3
ii	4	III	-5.4E-4	-3.9E-4	-2.1E-4	-6.9E-5	2.20E-3	2.07E-3	2.07E-3	1.96E-3
ii	5	I	-5.2E-2	6.5E-3	-2.6E-2	-3.4E-5	7.83E-2	5.22E-2	5.22E-2	4.39E-2
ii	5	II	-1.7E-3	-3.8E-4	-1.4E-3	-8.1E-6	6.59E-3	5.91E-3	5.91E-3	6.61E-3
ii	5	III	-5.4E-4	-3.8E-4	-9.3E-5	5.4E-5	2.23E-3	2.09E-3	2.09E-3	2.00E-3
iii	1	I	-4.1E-3	-5.0E-4	-3.1E-3	5.8E-5	1.09E-2	8.05E-3	8.05E-3	9.20E-3
iii	1	II	-4.5E-3	-1.7E-4	-4.3E-3	-2.2E-5	1.19E-2	9.01E-3	9.01E-3	1.22E-2
iii	1	III	-5.4E-3	5.1E-4	-5.9E-3	-2.0E-4	1.41E-2	1.04E-2	1.04E-2	1.58E-2
iii	2	I	-2.3E-3	-8.2E-4	-1.1E-3	-2.7E-4	9.29E-3	8.76E-3	8.76E-3	5.29E-3
iii	2	II	-2.5E-3	-8.9E-4	-1.7E-3	-4.7E-4	9.62E-3	8.94E-3	8.94E-3	6.55E-3
iii	2	III	-2.6E-3	-5.3E-4	-2.4E-3	-6.4E-4	1.03E-2	9.74E-3	9.74E-3	8.62E-3
iii	3	I	-2.3E-3	-8.2E-4	-9.5E-4	-1.8E-4	9.00E-3	8.48E-3	8.48E-3	4.38E-3
iii	3	II	-2.5E-3	-8.8E-4	-1.4E-3	-3.5E-4	9.22E-3	8.56E-3	8.56E-3	5.38E-3
iii	3	III	-2.5E-3	-5.5E-4	-2.0E-3	-5.0E-4	9.62E-3	9.04E-3	9.04E-3	7.00E-3
iii	4	I	-2.7E-1	1.2E-2	-1.3E-1	-1.2E-4	3.80E-1	9.70E-2	9.70E-2	1.88E-1
iii	4	II	-7.6E-3	-6.8E-4	-7.3E-3	-2.6E-4	1.77E-2	1.01E-2	1.01E-2	1.73E-2
iii	4	III	-1.5E-3	-6.8E-4	-8.6E-4	-1.2E-4	4.49E-3	3.59E-3	3.59E-3	3.69E-3
iii	5	I	-2.7E-1	1.1E-2	-1.3E-1	9.2E-5	3.81E-1	9.34E-2	9.34E-2	1.88E-1
iii	5	II	-7.7E-3	-6.5E-4	-7.3E-3	-1.3E-5	1.80E-2	1.02E-2	1.02E-2	1.73E-2
iii	5	III	-1.5E-3	-6.6E-4	-6.9E-4	9.3E-5	4.54E-3	3.61E-3	3.61E-3	3.64E-3

**Table 3:** Bias and *RMSE* for the parameter estimates for  $\rho = 0.5$ . Cross-sectional dimension  $n = 5$ , time series dimension  $T = 200$ .  $M = 1000$  Monte Carlo steps.

## 5.1 Data

In this analysis CDS spreads are used to describe the implied credit risk of a firm.<sup>8</sup> The insurance premium the buyer has to pay to the seller is the CDS premium. The CDS premium is the amount payable per year to insure against the event of default of any underlying with notational amount 1; it is usually measured in basis points. With the usual quarterly frequency, the buyer pays  $premium/(4 \cdot 10000)$  times the nominal value stipulated in the contract to the seller. The probability of default and the loss given default (one

<sup>8</sup> With a CDS contract a protection buyer acquires insurance against the default of a specified company. The protection seller declares his willingness to compensate the protection buyer for a loss arising in the case of default of the specified entity. For more details on the specification of credit default swap contracts we refer the reader to the International Securities and Derivatives Association (ISDA); [www.isda.org](http://www.isda.org).

W	DGP	$\Sigma_\epsilon$	Bias				RMSE			
			OLS	2SLS	DOLS	D2SLS	OLS	2SLS	DOLS	D2SLS
iv	1	I	-5.6E-4	3.0E-5	-4.5E-4	2.7E-5	2.57E-3	2.42E-3	2.42E-3	2.27E-3
iv	1	II	-5.9E-4	6.3E-5	-5.6E-4	7.5E-5	2.67E-3	2.54E-3	2.54E-3	2.68E-3
iv	1	III	-6.1E-4	1.3E-4	-6.2E-4	1.5E-4	2.90E-3	2.79E-3	2.79E-3	3.11E-3
iv	2	I	-4.2E-4	-1.4E-4	-2.5E-4	-9.6E-5	2.93E-3	3.02E-3	3.02E-3	1.46E-3
iv	2	II	-4.9E-4	-2.1E-4	-3.5E-4	-1.4E-4	3.07E-3	3.13E-3	3.13E-3	1.89E-3
iv	2	III	-5.0E-4	-1.6E-4	-4.7E-4	-1.7E-4	3.27E-3	3.32E-3	3.32E-3	2.57E-3
iv	3	I	-4.0E-4	-1.3E-4	-2.1E-4	-7.8E-5	2.87E-3	2.96E-3	2.96E-3	1.21E-3
iv	3	II	-4.6E-4	-1.9E-4	-2.8E-4	-1.1E-4	2.96E-3	3.01E-3	3.01E-3	1.53E-3
iv	3	III	-4.8E-4	-1.5E-4	-4.0E-4	-1.4E-4	3.08E-3	3.12E-3	3.12E-3	2.08E-3
iv	4	I	-4.5E-2	-1.6E-3	-2.1E-2	-8.4E-4	6.40E-2	2.20E-2	2.20E-2	3.22E-2
iv	4	II	-9.7E-4	-1.9E-4	-9.2E-4	-1.4E-4	2.92E-3	2.47E-3	2.47E-3	2.92E-3
iv	4	III	-1.9E-4	-6.3E-5	-1.3E-4	-4.4E-5	8.76E-4	8.35E-4	8.35E-4	8.20E-4
iv	5	I	-4.5E-2	-1.6E-3	-2.2E-2	-8.7E-4	6.42E-2	2.17E-2	2.17E-2	3.24E-2
iv	5	II	-9.6E-4	-1.8E-4	-9.1E-4	-1.4E-4	2.88E-3	2.45E-3	2.45E-3	2.92E-3
iv	5	III	-1.8E-4	-6.0E-5	-1.2E-4	-4.4E-5	8.68E-4	8.28E-4	8.28E-4	8.21E-4
v	1	I	-7.5E-4	-1.6E-4	-5.4E-4	-5.0E-5	3.49E-3	3.28E-3	3.28E-3	3.07E-3
v	1	II	-7.4E-4	-5.6E-5	-6.6E-4	1.2E-5	3.78E-3	3.60E-3	3.60E-3	3.77E-3
v	1	III	-6.3E-4	1.7E-4	-7.2E-4	1.0E-4	4.28E-3	4.12E-3	4.12E-3	4.62E-3
v	2	I	-5.2E-4	-2.6E-4	-3.2E-4	-1.8E-4	4.06E-3	4.21E-3	4.21E-3	2.09E-3
v	2	II	-5.9E-4	-3.0E-4	-4.4E-4	-2.4E-4	4.23E-3	4.35E-3	4.35E-3	2.62E-3
v	2	III	-5.5E-4	-1.8E-4	-5.7E-4	-2.5E-4	4.46E-3	4.56E-3	4.56E-3	3.51E-3
v	3	I	-5.0E-4	-2.4E-4	-2.6E-4	-1.4E-4	3.97E-3	4.12E-3	4.12E-3	1.72E-3
v	3	II	-5.6E-4	-2.8E-4	-3.6E-4	-1.8E-4	4.08E-3	4.21E-3	4.21E-3	2.13E-3
v	3	III	-5.4E-4	-1.8E-4	-4.7E-4	-2.1E-4	4.20E-3	4.31E-3	4.31E-3	2.84E-3
v	4	I	-4.7E-2	-2.4E-4	-2.2E-2	-1.0E-3	6.81E-2	2.98E-2	2.98E-2	3.70E-2
v	4	II	-1.2E-3	-3.2E-4	-1.0E-3	-1.8E-4	3.91E-3	3.46E-3	3.46E-3	3.88E-3
v	4	III	-3.0E-4	-1.7E-4	-1.6E-4	-6.7E-5	1.27E-3	1.22E-3	1.22E-3	1.14E-3
v	5	I	-4.7E-2	-3.1E-4	-2.3E-2	-1.0E-3	6.84E-2	2.92E-2	2.92E-2	3.66E-2
v	5	II	-1.2E-3	-3.1E-4	-1.0E-3	-1.5E-4	3.86E-3	3.43E-3	3.43E-3	3.85E-3
v	5	III	-2.9E-4	-1.7E-4	-1.2E-4	-3.5E-5	1.26E-3	1.21E-3	1.21E-3	1.14E-3

**Table 4:** Bias and *RMSE* for the parameter estimates for  $\rho = 0.5$ . Cross-sectional dimension  $n = 5$ , time series dimension  $T = 200$ .  $M = 1000$  Monte Carlo steps.

minus the recovery rate) should be the main driving forces of the CDS spreads (see e.g. Hull (2006), Schönbucher (2003)).

We utilize the dataset already used in Schneider et al. (2010), where CDS spreads of 278 firms – obtained from the *Markit Group* – have been investigated. We focus on the five year maturities which are said to be the most liquid ones (see e.g. Hull et al. (2004)). The observation period is January 2, 2001 to May 30, 2008. In line with a bulk of quantitative finance literature we stick to weekly data, such that  $T = 230$ . Using weekly data instead of daily observations is often done to avoid day of the week effects.

Next the CDS data are matched with firm specific characteristics obtained from *Thomson Datastream* and *Compustat* data. We construct the KMV distance to default,  $DD_{it}$  from firm specific data by following Crosbie and Bohn (2003). Moreover, we calculate the debt to value ratio,  $DVR_{it}$ . This firm specific data was available for 176 out of the 278 firms. Following Berndt et al. (2008) we also include the *VIX* volatility index from the *Chicago Board Options Exchange* (<http://www.cboe.com/micro/VIX/vixintro.aspx>) as an explanatory variable. Additionally, we include a short run and a long run interest rate obtained from the *Federal Reserve* (<http://federalreserve.gov/releases/h15/data.htm>). In more details we use

<i>W</i>	DGP	$\Sigma_\epsilon$	$\alpha_c = 0.01$				$\alpha_c = 0.05$				$\alpha_c = 0.10$			
			<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>	<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>	<i>OLS</i>	<i>2SLS</i>	<i>DOLS</i>	<i>D2SLS</i>
i	1	I	1.3	1.3	1.9	1.9	5.3	5.6	8.8	9.2	10.9	10.9	14.2	14.0
i	1	II	1.4	1.4	2.2	2.4	6.5	6.5	8.9	8.8	11.6	11.7	15.8	15.8
i	1	III	1.4	1.4	2.9	3.0	6.6	6.6	9.5	9.3	11.4	11.4	14.5	14.6
i	2	I	1.2	1.1	0.8	0.9	5.1	5.2	4.7	4.9	9.7	9.6	10.2	10.4
i	2	II	1.0	1.1	2.1	2.3	5.4	5.5	8.2	8.3	10.5	10.4	14.1	13.8
i	2	III	1.6	1.7	4.0	3.9	6.7	6.8	10.3	10.2	13.5	13.7	16.6	16.8
i	3	I	1.1	1.1	0.4	0.5	4.9	4.9	4.0	4.0	9.8	9.5	7.9	7.9
i	3	II	0.9	1.0	1.2	1.2	5.2	5.3	7.1	7.3	10.2	10.4	12.2	12.5
i	3	III	1.6	1.6	3.3	3.2	6.7	6.5	9.8	9.9	13.5	13.5	15.6	15.5
i	4	I	0.2	0.2	0.8	0.8	3.2	3.0	6.2	6.0	7.8	6.8	12.4	12.2
i	4	II	0.0	0.0	0.6	0.8	3.2	3.2	5.8	6.0	8.4	8.4	11.8	11.8
i	4	III	0.4	0.4	1.0	1.0	4.0	4.0	6.4	6.4	8.8	8.8	12.0	12.0
i	5	I	2.4	2.0	5.6	6.2	8.2	7.6	14.2	14.2	14.4	13.2	21.0	20.8
i	5	II	2.0	2.0	4.8	4.6	8.4	8.6	13.4	12.4	15.6	15.4	21.0	21.2
i	5	III	2.2	2.2	4.0	4.0	8.2	8.2	11.8	11.8	15.4	15.4	20.0	20.2
ii	1	I	1.3	1.3	2.0	2.1	5.6	5.5	8.2	8.2	11.2	11.0	16.0	16.0
ii	1	II	1.3	1.2	1.7	1.5	6.3	6.3	9.5	9.4	11.2	11.3	15.9	16.1
ii	1	III	1.2	1.2	1.9	2.0	6.3	6.0	9.4	9.7	11.9	12.0	16.2	16.4
ii	2	I	0.8	0.8	1.1	1.0	4.1	4.1	4.9	4.7	8.5	8.4	10.4	10.6
ii	2	II	1.4	1.4	1.8	1.8	4.3	4.3	7.0	7.0	9.0	9.4	13.8	13.8
ii	2	III	1.8	1.8	2.5	2.4	6.4	6.6	10.0	9.8	11.2	11.0	15.1	15.6
ii	3	I	0.9	0.9	0.9	0.9	3.8	3.9	3.8	3.8	8.3	8.3	8.2	8.0
ii	3	II	1.3	1.2	2.0	2.0	4.6	4.4	6.4	6.3	8.4	8.6	12.2	12.1
ii	3	III	1.6	1.6	2.2	2.2	6.5	6.5	8.8	9.0	11.3	11.2	14.4	14.8
ii	4	I	0.0	0.4	0.6	1.2	4.0	3.8	6.0	5.0	7.8	7.2	9.6	9.8
ii	4	II	0.2	0.2	0.4	0.4	3.8	3.8	6.0	5.6	7.8	7.8	10.4	10.2
ii	4	III	0.2	0.2	0.6	0.6	4.6	4.6	5.6	5.6	7.8	8.0	11.2	11.2
ii	5	I	3.6	2.8	4.8	4.6	9.2	8.0	12.8	13.4	16.0	13.6	22.8	22.2
ii	5	II	3.6	3.4	5.0	4.6	8.4	8.6	11.2	11.0	15.4	15.0	19.0	18.8
ii	5	III	3.0	3.0	4.4	4.4	9.2	9.0	11.2	11.0	16.0	15.8	20.0	20.2
iii	1	I	1.6	1.7	2.4	2.6	5.6	5.7	8.0	7.9	13.0	13.0	12.9	12.9
iii	1	II	1.3	1.3	2.3	2.4	6.6	6.7	7.5	7.6	12.9	12.7	13.7	13.8
iii	1	III	1.2	1.3	2.2	2.3	6.8	6.6	9.3	9.0	12.4	12.3	13.7	13.6
iii	2	I	1.0	0.9	1.5	1.5	4.0	4.2	4.2	4.2	8.0	7.9	8.9	8.9
iii	2	II	1.4	1.3	2.3	2.4	5.1	5.1	8.0	7.9	10.0	9.8	13.7	13.5
iii	2	III	1.6	1.6	3.3	3.3	6.3	6.6	9.9	9.9	12.2	12.1	15.1	15.2
iii	3	I	0.9	1.0	1.0	0.9	4.3	4.2	3.7	3.6	8.0	8.0	7.0	7.0
iii	3	II	1.1	1.2	1.8	1.9	5.3	5.1	7.3	7.4	9.9	10.4	11.9	12.1
iii	3	III	1.4	1.4	2.8	2.8	7.0	6.8	8.7	8.5	12.5	12.6	15.4	15.3
iii	4	I	0.6	0.6	2.2	2.4	4.0	3.6	6.0	5.0	9.0	7.8	12.0	10.8
iii	4	II	0.6	0.6	1.0	1.0	4.2	4.2	6.6	6.6	9.8	9.6	12.4	12.4
iii	4	III	0.4	0.4	1.0	1.0	4.6	4.6	7.2	7.2	9.0	9.0	11.8	11.8
iii	5	I	3.6	2.6	4.8	5.0	10.2	10.0	13.6	13.6	16.4	15.8	23.2	21.2
iii	5	II	3.0	3.0	5.6	5.2	10.8	11.2	14.0	13.6	18.2	18.6	22.6	22.2
iii	5	III	3.2	3.2	5.2	5.2	9.8	9.8	14.4	14.4	17.4	17.6	22.4	22.4

**Table 5:** Size for the parameter  $\rho$ : Rejections rates of the Wald test for the true null hypothesis  $\rho = 0$  in percentage terms, given the significance levels  $\alpha_c = \{0.01, 0.05, 0.1\}$ .  $M = 1000$  simulation runs. Cross-sectional dimension  $n = 50$ , time series dimension  $T = 200$ .

two year and ten year US treasury yields, denoted by  $r_{2t}$  and  $r_{10t}$ , respectively. Since a firm's cost of capital is usually affected by interest rates, government bond yields are often included when credit risk is investigated. A more detailed description of the data and the construction of the explanatory variables is



$W$	Des.	$\Sigma_\varepsilon$	$\alpha_c = 0.01$				$\alpha_c = 0.05$				$\alpha_c = 0.10$			
			$OLS$	$2SLS$	$DOLS$	$D2SLS$	$OLS$	$2SLS$	$DOLS$	$D2SLS$	$OLS$	$2SLS$	$DOLS$	$D2SLS$
iv	1	I	1.3	1.4	1.7	1.7	5.5	5.4	8.6	8.6	11.0	10.7	15.2	15.0
iv	1	II	1.6	1.5	2.3	2.4	6.4	6.4	9.1	9.1	12.1	12.1	15.9	15.8
iv	1	III	1.5	1.4	3.3	3.1	6.9	6.9	8.7	8.9	11.7	11.6	15.1	15.0
iv	2	I	1.0	1.0	0.8	0.8	5.2	5.1	5.0	4.8	9.5	9.5	9.8	9.6
iv	2	II	1.3	1.2	1.9	2.0	5.2	5.3	7.5	7.6	10.4	10.2	13.9	13.8
iv	2	III	1.6	1.6	3.5	3.7	6.9	7.0	10.1	9.8	13.0	13.1	16.4	16.4
iv	3	I	1.0	1.0	0.5	0.5	4.7	4.7	3.6	3.6	9.5	9.5	8.2	8.2
iv	3	II	1.1	1.1	1.2	1.1	5.0	4.9	6.8	6.6	10.6	10.6	12.2	12.0
iv	3	III	1.6	1.6	3.2	3.2	6.5	6.5	8.9	9.0	12.9	12.9	15.7	15.4
iv	4	I	0.2	0.2	0.8	0.8	3.6	3.8	5.8	5.8	7.4	7.0	11.4	11.6
iv	4	II	0.2	0.2	1.0	1.0	3.8	3.8	7.0	7.0	7.8	8.0	12.8	12.8
iv	4	III	0.6	0.6	1.2	1.2	4.2	4.2	6.8	6.8	8.8	8.8	13.2	13.0
iv	5	I	2.6	2.4	5.6	5.8	8.6	7.6	13.8	14.4	14.2	15.2	22.2	22.4
iv	5	II	2.4	2.4	4.8	4.8	8.2	8.4	13.2	13.2	16.4	16.2	20.8	20.8
iv	5	III	2.2	2.2	3.8	3.8	9.0	9.0	12.6	12.6	15.6	15.4	19.2	19.6
v	1	I	1.3	1.3	1.7	1.6	4.5	4.6	7.9	7.7	10.4	10.3	14.0	13.9
v	1	II	1.2	1.2	2.1	2.1	4.7	4.7	7.7	7.5	10.7	10.5	14.9	14.6
v	1	III	1.1	1.2	2.1	2.1	5.4	5.4	8.0	7.9	10.7	10.6	15.3	15.1
v	2	I	0.7	0.7	0.8	0.9	4.6	4.8	3.9	3.9	9.4	9.4	9.1	9.0
v	2	II	1.0	0.9	1.6	1.6	5.0	5.1	7.2	7.5	10.9	11.1	13.6	13.4
v	2	III	1.8	1.6	3.1	3.1	6.9	7.0	10.5	10.4	11.3	11.4	16.5	17.0
v	3	I	0.6	0.6	0.6	0.6	4.7	4.7	2.8	2.8	9.5	9.3	6.9	7.1
v	3	II	0.9	1.0	1.3	1.3	4.6	4.8	6.5	6.4	10.3	10.3	12.3	12.1
v	3	III	2.0	1.9	2.6	2.6	7.0	7.0	9.4	9.4	11.0	11.1	16.5	16.4
v	4	I	0.0	0.2	1.4	1.4	3.4	4.0	6.4	5.4	8.0	6.8	10.4	11.0
v	4	II	0.4	0.4	2.0	2.0	4.8	5.0	6.0	6.2	7.6	7.6	13.0	12.6
v	4	III	0.2	0.2	2.0	2.0	5.0	5.0	6.4	6.4	8.4	8.2	11.6	11.6
v	5	I	1.0	1.7	4.0	4.0	6.0	4.7	10.3	10.0	11.7	9.0	20.0	18.0
v	5	II	2.3	2.3	4.3	4.3	5.7	5.7	10.3	10.0	12.3	12.3	17.0	17.3
v	5	III	2.0	2.0	3.7	3.7	6.7	6.7	10.3	10.3	13.0	12.3	17.7	18.0

**Table 6:** Size for the parameter  $\rho$ : Rejections rates of the Wald test for the true null hypothesis  $\rho = 0$  in percentage terms, given the significance levels  $\alpha_c = \{0.01, 0.05, 0.1\}$ .  $M = 1000$  simulation runs. Cross-sectional dimension  $n = 5$ , time series dimension  $T = 200$ .

provided in Appendix C.

To apply and estimate the spatial autocorrelation model the spatial weights matrix  $\mathbf{W}$  has to be constructed. We use the industry-by-industry total requirements matrix for the year 2002 provided by the *Bureau of Labor Statistics* (BLS) and match each firm in our data to a particular BLS industry. In this data set the total requirements matrix contains for each industry the proportion of inputs ultimately stemming from each other industry relative to its own sales. We use this to proxy for possible correlation of shocks coming through the supply chain. The resulting weights matrix thus approximates the possible correlation patterns due to technology and demand shocks working their way through the economy. The elements along the main diagonal are set to zero. To improve the numerical properties and to be able to interpret the estimated coefficients, we normalize our spatial weights matrix by its largest absolute

W	DGP	$\Sigma_\epsilon$	Bias				RMSE			
			OLS	2SLS	DOLS	D2SLS	OLS	2SLS	DOLS	D2SLS
i	1	I	-3.1E-5	-2.1E-6	-2.8E-5	-3.4E-6	6.96E-5	8.49E-5	8.49E-5	6.00E-5
i	1	II	-3.4E-5	-2.2E-6	-3.5E-5	-4.1E-6	7.54E-5	9.02E-5	9.02E-5	7.29E-5
i	1	III	-3.9E-5	-1.4E-6	-3.9E-5	-3.1E-6	8.10E-5	9.51E-5	9.51E-5	8.52E-5
i	2	I	-3.6E-5	-8.6E-6	-1.5E-5	-6.9E-7	1.19E-4	1.72E-4	1.72E-4	5.36E-5
i	2	II	-3.8E-5	-5.1E-6	-2.1E-5	1.4E-8	1.23E-4	1.74E-4	1.74E-4	6.80E-5
i	2	III	-3.9E-5	1.3E-6	-2.9E-5	6.3E-7	1.25E-4	1.74E-4	1.74E-4	8.86E-5
i	3	I	-3.5E-5	-8.5E-6	-1.3E-5	-9.2E-7	1.18E-4	1.70E-4	1.70E-4	4.52E-5
i	3	II	-3.7E-5	-5.2E-6	-1.8E-5	-2.0E-7	1.20E-4	1.71E-4	1.71E-4	5.64E-5
i	3	III	-3.7E-5	1.2E-6	-2.4E-5	6.1E-7	1.21E-4	1.68E-4	1.68E-4	7.24E-5
i	4	I	-2.7E-3	-2.1E-5	-1.3E-3	-2.9E-5	3.39E-3	6.25E-4	6.25E-4	1.65E-3
i	4	II	-3.3E-5	-2.5E-6	-3.3E-5	-2.9E-6	6.14E-5	6.52E-5	6.52E-5	6.44E-5
i	4	III	-4.7E-6	-7.2E-7	-3.9E-6	-8.1E-7	1.64E-5	2.20E-5	2.20E-5	1.65E-5
i	5	I	-2.7E-3	-2.3E-5	-1.3E-3	-3.1E-5	3.40E-3	6.22E-4	6.22E-4	1.65E-3
i	5	II	-3.3E-5	-2.6E-6	-3.3E-5	-2.4E-6	6.17E-5	6.53E-5	6.53E-5	6.41E-5
i	5	III	-4.8E-6	-7.5E-7	-3.7E-6	-6.8E-7	1.65E-5	2.21E-5	2.21E-5	1.65E-5
ii	1	I	-4.0E-5	-4.5E-6	-3.5E-5	-6.8E-6	1.12E-4	1.25E-4	1.25E-4	9.73E-5
ii	1	II	-4.2E-5	-2.0E-6	-4.2E-5	-5.9E-6	1.18E-4	1.32E-4	1.32E-4	1.12E-4
ii	1	III	-4.4E-5	3.2E-6	-4.6E-5	-2.1E-6	1.20E-4	1.33E-4	1.33E-4	1.26E-4
ii	2	I	-4.1E-5	-1.8E-5	-1.5E-5	-1.1E-8	1.73E-4	2.29E-4	2.29E-4	8.23E-5
ii	2	II	-4.6E-5	-1.7E-5	-2.1E-5	3.8E-7	1.82E-4	2.31E-4	2.31E-4	1.04E-4
ii	2	III	-4.4E-5	-6.0E-6	-3.1E-5	2.5E-6	1.85E-4	2.31E-4	2.31E-4	1.35E-4
ii	3	I	-4.1E-5	-1.8E-5	-1.3E-5	-1.6E-7	1.71E-4	2.27E-4	2.27E-4	6.93E-5
ii	3	II	-4.6E-5	-1.8E-5	-1.8E-5	2.4E-8	1.79E-4	2.28E-4	2.28E-4	8.56E-5
ii	3	III	-4.4E-5	-6.6E-6	-2.6E-5	2.1E-6	1.79E-4	2.23E-4	2.23E-4	1.10E-4
ii	4	I	-3.1E-3	-3.2E-5	-1.4E-3	-6.7E-5	4.29E-3	8.83E-4	8.83E-4	2.12E-3
ii	4	II	-3.9E-5	-5.0E-6	-3.8E-5	-6.1E-6	9.11E-5	9.20E-5	9.20E-5	9.37E-5
ii	4	III	-6.5E-6	-1.6E-6	-4.3E-6	-1.7E-6	2.64E-5	3.13E-5	3.13E-5	2.61E-5
ii	5	I	-3.1E-3	-4.2E-5	-1.4E-3	-6.6E-5	4.31E-3	8.79E-4	8.79E-4	2.10E-3
ii	5	II	-4.0E-5	-6.0E-6	-3.7E-5	-5.3E-6	9.19E-5	9.19E-5	9.19E-5	9.25E-5
ii	5	III	-6.8E-6	-2.0E-6	-3.6E-6	-1.4E-6	2.64E-5	3.16E-5	3.16E-5	2.59E-5
iii	1	I	-5.3E-5	-9.0E-6	-4.6E-5	-7.1E-6	1.64E-4	1.71E-4	1.71E-4	1.41E-4
iii	1	II	-5.6E-5	-8.9E-6	-5.5E-5	-8.0E-6	1.73E-4	1.83E-4	1.83E-4	1.63E-4
iii	1	III	-5.7E-5	-4.4E-6	-5.9E-5	-5.5E-6	1.71E-4	1.85E-4	1.85E-4	1.77E-4
iii	2	I	-4.6E-5	-2.1E-5	-1.5E-5	-8.4E-7	2.12E-4	2.84E-4	2.84E-4	1.08E-4
iii	2	II	-5.2E-5	-2.0E-5	-2.3E-5	1.5E-6	2.28E-4	2.99E-4	2.99E-4	1.39E-4
iii	2	III	-4.6E-5	-5.2E-6	-3.4E-5	-8.4E-7	2.32E-4	3.20E-4	3.20E-4	1.76E-4
iii	3	I	-4.7E-5	-2.2E-5	-1.3E-5	-7.8E-7	2.10E-4	2.80E-4	2.80E-4	9.01E-5
iii	3	II	-5.3E-5	-2.1E-5	-1.9E-5	1.4E-6	2.23E-4	2.94E-4	2.94E-4	1.14E-4
iii	3	III	-4.6E-5	-6.5E-6	-2.9E-5	-4.0E-7	2.24E-4	3.09E-4	3.09E-4	1.44E-4
iii	4	I	-3.6E-3	-4.9E-5	-1.7E-3	-7.0E-5	5.45E-3	1.27E-3	1.27E-3	2.68E-3
iii	4	II	-5.0E-5	-1.1E-5	-4.6E-5	-8.9E-6	1.29E-4	1.38E-4	1.38E-4	1.30E-4
iii	4	III	-9.3E-6	-4.4E-6	-4.9E-6	-2.8E-6	3.78E-5	4.64E-5	4.64E-5	3.63E-5
iii	5	I	-3.6E-3	-6.5E-5	-1.7E-3	-7.2E-5	5.46E-3	1.26E-3	1.26E-3	2.64E-3
iii	5	II	-5.1E-5	-1.3E-5	-4.5E-5	-8.0E-6	1.32E-4	1.38E-4	1.38E-4	1.27E-4
iii	5	III	-9.6E-6	-4.9E-6	-3.7E-6	-2.2E-6	3.82E-5	4.68E-5	4.68E-5	3.59E-5

**Table 7:** Bias and *RMSE* for the parameter estimates for  $\rho = 0.95$ . Cross-sectional dimension  $n = 50$ , time series dimension  $T = 200$ .  $M = 1000$  Monte Carlo steps.

eigenvalue. As a result, the spatial autocorrelation parameter in our model is bounded by one from above and thus has the usual range as in the time series autocorrelation models. I.e. with the zeros in the main diagonal and  $\rho \in (-1, 1)$ , the requirements of Assumption 1 are fulfilled.  $\rho$  can be interpreted like a usual correlation parameter.

After matching the CDS data with the data collected from *Thomson Datastream*, *Compustat* and the *Bureau of Labor Statistics* and correcting for firms where we detected problems in data (e.g. extreme spikes, missing values, unclear industry affiliation, etc) we arrived at a cross-section of  $N = 148$  firms. A clustering of the data toward the first digit of the NAICs industry classification and the S&P rating results in Table 9. A NAICs code starting with 2 stands for mining, utilities or construction, 3 for manufacturing,

W	DGP	$\Sigma_\epsilon$	Bias				RMSE			
			OLS	2SLS	DOLS	D2SLS	OLS	2SLS	DOLS	D2SLS
iv	1	I	-3.2E-5	-2.8E-6	-2.9E-5	-3.9E-6	7.52E-5	8.86E-5	8.86E-5	6.35E-5
iv	1	II	-3.5E-5	-2.9E-6	-3.7E-5	-4.7E-6	8.05E-5	9.38E-5	9.38E-5	7.66E-5
iv	1	III	-4.0E-5	-2.2E-6	-4.1E-5	-3.9E-6	8.46E-5	9.98E-5	9.98E-5	8.89E-5
iv	2	I	-3.7E-5	-9.1E-6	-1.6E-5	-6.4E-7	1.28E-4	1.80E-4	1.80E-4	5.63E-5
iv	2	II	-3.9E-5	-5.8E-6	-2.2E-5	-2.8E-7	1.30E-4	1.81E-4	1.81E-4	7.15E-5
iv	2	III	-4.0E-5	3.2E-7	-3.0E-5	1.1E-8	1.31E-4	1.82E-4	1.82E-4	9.20E-5
iv	3	I	-3.6E-5	-9.1E-6	-1.4E-5	-9.2E-7	1.27E-4	1.78E-4	1.78E-4	4.76E-5
iv	3	II	-3.8E-5	-5.9E-6	-1.9E-5	-4.7E-7	1.28E-4	1.78E-4	1.78E-4	5.93E-5
iv	3	III	-3.8E-5	2.3E-7	-2.5E-5	1.3E-7	1.26E-4	1.76E-4	1.76E-4	7.52E-5
iv	4	I	-2.9E-3	-2.8E-5	-1.4E-3	-2.7E-5	3.64E-3	6.75E-4	6.75E-4	1.76E-3
iv	4	II	-3.6E-5	-3.5E-6	-3.6E-5	-3.7E-6	6.58E-5	6.98E-5	6.98E-5	6.87E-5
iv	4	III	-5.3E-6	-1.1E-6	-4.4E-6	-1.1E-6	1.74E-5	2.35E-5	2.35E-5	1.73E-5
iv	5	I	-2.9E-3	-3.1E-5	-1.4E-3	-2.9E-5	3.65E-3	6.73E-4	6.73E-4	1.76E-3
iv	5	II	-3.6E-5	-3.7E-6	-3.6E-5	-3.2E-6	6.60E-5	6.98E-5	6.98E-5	6.82E-5
iv	5	III	-5.4E-6	-1.2E-6	-4.1E-6	-9.4E-7	1.75E-5	2.35E-5	2.35E-5	1.73E-5
v	1	I	-3.6E-5	-4.4E-6	-3.2E-5	-6.5E-6	9.37E-5	1.06E-4	1.06E-4	7.87E-5
v	1	II	-3.9E-5	-5.3E-6	-4.0E-5	-8.3E-6	9.92E-5	1.11E-4	1.11E-4	9.35E-5
v	1	III	-4.2E-5	-4.3E-6	-4.3E-5	-7.3E-6	1.02E-4	1.17E-4	1.17E-4	1.08E-4
v	2	I	-3.9E-5	-1.1E-5	-1.7E-5	1.3E-6	1.57E-4	2.00E-4	2.00E-4	7.08E-5
v	2	II	-4.2E-5	-8.8E-6	-2.3E-5	2.3E-6	1.62E-4	2.00E-4	2.00E-4	8.97E-5
v	2	III	-4.2E-5	-3.4E-6	-3.2E-5	3.5E-6	1.63E-4	2.02E-4	2.02E-4	1.17E-4
v	3	I	-3.9E-5	-1.2E-5	-1.4E-5	6.2E-7	1.55E-4	1.98E-4	1.98E-4	5.99E-5
v	3	II	-4.1E-5	-9.6E-6	-2.0E-5	1.3E-6	1.59E-4	1.96E-4	1.96E-4	7.43E-5
v	3	III	-4.1E-5	-4.2E-6	-2.7E-5	2.9E-6	1.57E-4	1.94E-4	1.94E-4	9.52E-5
v	4	I	-3.1E-3	-3.0E-6	-1.5E-3	-1.8E-5	4.06E-3	7.96E-4	7.96E-4	1.98E-3
v	4	II	-3.9E-5	-2.3E-6	-3.9E-5	-2.5E-6	8.13E-5	8.20E-5	8.20E-5	8.37E-5
v	4	III	-6.1E-6	-9.0E-7	-4.6E-6	-7.0E-7	2.26E-5	2.75E-5	2.75E-5	2.21E-5
v	5	I	-3.0E-3	-5.1E-6	-1.4E-3	-1.9E-6	3.92E-3	7.04E-4	7.04E-4	1.86E-3
v	5	II	-3.7E-5	-2.5E-6	-3.6E-5	-1.8E-6	7.89E-5	7.41E-5	7.41E-5	7.90E-5
v	5	III	-6.1E-6	-1.1E-6	-4.1E-6	-6.7E-7	2.22E-5	2.53E-5	2.53E-5	2.09E-5

**Table 8:** Bias and *RMSE* for the parameter estimates for  $\rho = 0.95$ . Cross-sectional dimension  $n = 50$ , time series dimension  $T = 200$ .  $M = 1000$  Monte Carlo steps.

4 for trade and transportation, 5 for information, banking and finance, 6 for educational services, health care and social assistance, while 7 stands for arts, entertainment, and accommodation and food services. For more details see <http://www.naics.com>.

Before we proceed with the econometric model, let us briefly discuss the expected impacts (expected based on economic theory, intuition and literature). The reader should note that the CDS spread is often used as an indicator for the probability of default of a firm. Since the distance to default measures the

	2	3	4	5	6	7	Total
AA	0	4	1	5	0	0	10
A	4	19	5	16	0	1	45
BBB	16	25	18	8	1	4	72
BB	1	6	6	1	0	0	14
B	1	3	2	0	1	0	7
Total	22	57	32	30	2	5	148

**Table 9:** Distribution of firms according to industry and rating. Horizontally first digit of the firm's NAIC Code. Cross-sectional dimension  $N = 148$ .

distance to the default boundary, we expect a lower spread if the distance to default increases. A raise in the firm's leverage should increase the default probability and therefore the CDS spread. With the interest rate the effect is not so clear. If the interest rate increases the cost of capital increases for a leveraged firm. This should drive up the CDS spread. With the volatility measure  $VIX$  we expect higher spreads in periods of higher volatility. The rating of a firm should also reflect the probability of default. Rating effects should be included in the fixed effects  $\alpha_i, i = 1, \dots, n$  in model (1). Last but not least we expect that CDS spreads are positively correlated, so we expect a positive  $\rho$  when (9) is estimated.

By Assumption 2 the explanatory variables  $\mathbf{x}_{it}$  should be  $I(1)$ . The question arises whether our model assumptions are compatible with the data observed. For the variables considered the distance to default should follow a geometric Brownian motion as long as the firm does not default based on the model assumptions (see e.g. Crosbie and Bohn (2003), Schönbucher (2003)). By construction  $DD_{it} \geq 0$ . Translated to discrete time the distance to default should follow a random walk with an absorbing barrier; only firms where this barrier is not hit are observed in the data set. The debt to value ratio lives on the interval  $[0, 100]$ , the  $VIX$  index measures volatility and is therefore be non-negative. Following applied literature and running augmented Dickey-Fuller tests on a unit root for these data show the following: The null of a unit root is not rejected for almost all time series on a five percent significance level for the CDS spreads. Im, Pesaran and Shin tests implemented in the *EViews* package provide us with the same results. For the distance to default the null of a unit root is rejected, although the serial correlation is quite high. For the debt-to-value ratios, the  $VIX$  and the interest rates there is stronger evidence for the presence of a unit root. Based on this discussion the data considered cannot exactly match the conditions of Assumption 2. Based on the time series properties we suppose that the model considered in Section 2 still provides us with a useful approximation of the (unknown) data generating process of the empirical data considered.

## 5.2 Results

Equipped with our data set we estimate the parameter vector  $\gamma$  by means of two-stage least squares, *DOLS*, *OLS* and *D2SLS*. The results are presented in Table 10. Based on the theoretical considerations

$\hat{\gamma}$	<i>OLS</i>		<i>2SLS</i>		<i>DOLS</i>		<i>D2SLS</i>	
$\rho$	0.5211	< 0.001	0.3920	0.0331	0.4936	< 0.001	0.3928	< 0.001
$\beta_{DVR}$	-14.6912	0.0092	-15.2656	0.4317	-21.5765	< 0.001	-22.2903	< 0.001
$\beta_{DD}$	5.3419	< 0.001	5.4718	< 0.001	5.1182	< 0.001	5.2172	< 0.001
$\beta_{r_2}$	4.6310	0.4589	4.2952	0.0025	9.9653	0.0256	9.9547	0.0260
$\beta_{r_{10}}$	-39.5642	0.0024	-42.4143	0.8117	-49.7624	< 0.001	-52.4572	< 0.001
$\beta_{VIX}$	-0.0472	0.8319	-0.0233	< 0.001	-0.1985	0.2076	-0.1869	0.2293

**Table 10: Parameter Estimates:** Model (9) applied to CDS data.  $y_{it}$  is CDS data on a firm level. The explanatory variables are the distance to default,  $DD_{it}$ , the debt to value ratio,  $DVR_{it}$ , a two year bond yield  $r_{2t}$ , a ten year bond yield  $r_{10t}$ , and the VIX volatility index  $VIX_t$ .  $T = 230$ ,  $N = 148$ ,  $p = 2$  and  $k_\rho = 2$ .

of the former chapters the *D2SLS* estimator should be used, the results from the other estimation methods are for comparison. When instrumental variables are used in the estimation, the debt-to-value ratio and the VIX are used in  $\sum_{i=1}^n \mathbf{W}\tilde{\mathbf{x}}_{it}$ , i.e.  $k_\rho = 2$ . For these two variables we observed the highest correlation with  $\sum_{i=1}^n \mathbf{W}\tilde{y}_{it}$ . All the p-values presented in Table 10 are obtained by means of a Wald test. For the distance to default and the debt to value ratio the parameters are highly significant, also the signs expected have been realized. Both interest rates are significant, where the short term interest rate  $r_{2t}$  increases the CDS spread, while the long term interest rate decreases the spread. In contrast to some results obtained in literature, the VIX volatility index is not significant when *D2SLS* estimation is performed and default significance levels (1%, 5%, 10%) are applied. The additional parameter which has been investigated in our analysis is the spatial correlation  $\rho$ . With the dynamic two stage least squares estimator the spatial correlation parameter  $\rho$  is positive as expected and highly significant. I.e. in addition to the methodological results obtained in the former sections, our model allows to include and to test for spatial correlation. Here we observed a significant effect.

## 6 Conclusions

In this paper we have studied panel data models with a cointegration relationship including a spatial lag. Due to this spatial lag standard estimation techniques do not provide us with appropriate tools to estimate the parameters and to perform inference. Based on this problem we stick to the usual assumptions used in the dynamic least squares estimation and develop a dynamic two stage least squares estimator. We show

that the parameter vector of interest is asymptotically independent of the nuisance parameters. Moreover, we derive the asymptotic distribution of the parameters, which also allows constructing a Wald test to perform statistical inference. Our estimation methodology is applied to simulated data to investigate the small sample properties and to financial data to test for the impact of spatial correlation on credit default swap spreads. Given this financial data set and a spatial correlation matrix obtained from input-output data, our analysis shows that spatial correlation is highly significant.

## A Proof of Theorem 1

The two stage least squares estimator is given by (22) and hence

$$\begin{aligned}
 (\widehat{\gamma}', \widehat{\delta}')'_{D2SLS} - (\gamma', \delta')' &= (\mathbf{X}'\mathbf{P}_H\mathbf{X})^{-1}\mathbf{X}'\mathbf{P}_H\mathbf{u}, \\
 &\text{where with } \mathbf{P}_H = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}' \text{ we get} \\
 &= (\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{u}.
 \end{aligned} \tag{36}$$

$\mathbf{X}$  is a  $Tn \times 1 + k + (2p + 1)k \cdot n$  matrix while  $\mathbf{Z}$  is of dimension  $Tn \times q_\rho + k + (2p + 1)k \cdot n$ . Note that the orthogonal projection  $\mathbf{P}_H$  on  $(\mathbf{x}_t, \zeta_t)$  is  $(\mathbf{x}_t, \zeta_t)$ . This yields

$$\mathbf{X}'\mathbf{P}_H = \begin{pmatrix} \mathbf{y}^* \\ \text{---} \\ \mathbf{x}' \\ \text{---} \\ \zeta' \end{pmatrix} \mathbf{P}_H = \begin{pmatrix} \mathbf{y}^*\mathbf{P}_H \\ \text{-----} \\ \mathbf{x}' \\ \text{-----} \\ \zeta' \end{pmatrix}. \tag{37}$$

Additionally,  $\mathbf{Z}_{it}$  is the transpose of the row of  $\mathbf{Z}$  corresponding to the index  $it$ . It is of dimension  $q_\rho + k + (2p + 1)kn \times 1$ .  $\mathbf{Z}_{it,1:d_z}$  consists of the first  $d_z$  elements of  $\mathbf{Z}_{it}$ , where  $d_z = q_\rho + k \times 1$ . The remaining elements of  $\mathbf{Z}_{it}$  contain  $\zeta_{it}$ . The "non- $it$  elements" of this vector are zero. In the same way we obtain  $\mathbf{X}_{it}$  which is of dimension  $1 + k + (2p + 1)kn \times 1$ . The first  $d_x$  elements are  $\mathbf{X}_{it,1:d_x}$ , where  $d_x = k + 1$ .  $d_x \leq d_z$  hold throughout the following analysis.

Due to Assumption 4 the leads and lags used in the projection are  $p = p(T)$ . Note that  $T_\star = T - 2p(T)$ , where  $T_\star \rightarrow \infty$  if  $T \rightarrow \infty$ . Therefore it is not necessary to distinguish between the  $T_\star \rightarrow \infty$  and  $T \rightarrow \infty$  when taking limits in the following analysis. In addition note that  $\underline{u}_{it}$  converges to  $u_{it}$  (this follows from Saikonen (1991)[Theorem 4.1/ Lemma A.5]).  $u_{it}$  is orthogonal to  $\zeta_{it}$ .

**Step 1:** Let us consider the term  $\mathbf{X}'\mathbf{P}_H\mathbf{X}$ . We normalize the elements of  $\mathbf{Z}$  and  $\mathbf{X}$  as follows: expand the first  $d_z$  and  $d_x$  elements by  $\frac{1}{T_\star}$ , the remaining terms (accounting for  $\zeta_t$ ) are multiplied by  $\frac{1}{\sqrt{T_\star}}$ . Based

on this we arrive at:

**Definition 1.** The  $d_x + (2p + 1)k \cdot n \times d_x + (2p + 1)k \cdot n$  matrix  $\mathbf{M}_{nT}^*$  is given by:

$$\begin{aligned} \mathbf{M}_{nT}^* = & \begin{pmatrix} \mathbf{y}'\mathbf{x}^*/(T_\star^2 n) & \mathbf{y}'\mathbf{x}/(T_\star^2 n) & \mathbf{y}'\zeta/(T_\star\sqrt{T_\star}) \\ \mathbf{x}'\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}'\mathbf{x}/(T_\star^2 n) & \mathbf{x}'\zeta/(T_\star\sqrt{T_\star}) \\ \zeta'\mathbf{x}^*/(T_\star\sqrt{T_\star}) & \zeta'\mathbf{x}/(T_\star\sqrt{T_\star}) & \zeta'\zeta/(\sqrt{T_\star}\sqrt{T_\star}) \end{pmatrix} \\ & \cdot \begin{pmatrix} \mathbf{x}^*\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}^*\mathbf{x}/(T_\star^2 n) & \mathbf{x}^*\zeta/(T_\star\sqrt{T_\star}) \\ \mathbf{x}'\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}'\mathbf{x}/(T_\star^2 n) & \mathbf{x}'\zeta/(T_\star\sqrt{T_\star}) \\ \zeta'\mathbf{x}^*/(T_\star\sqrt{T_\star}) & \zeta'\mathbf{x}/(T_\star\sqrt{T_\star}) & \zeta'\zeta/(\sqrt{T_\star}\sqrt{T_\star}) \end{pmatrix}^{-1} \\ & \cdot \begin{pmatrix} \mathbf{x}^*\mathbf{y}/(T_\star^2 n) & \mathbf{x}^*\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}^*\zeta'/(T_\star\sqrt{T_\star}) \\ \mathbf{x}'\mathbf{y}^*/(T_\star^2 n) & \mathbf{x}'\mathbf{x}/(T_\star^2 n) & \mathbf{x}'\zeta/(T_\star\sqrt{T_\star}) \\ \zeta'\mathbf{y}^*/(T_\star\sqrt{T_\star}) & \zeta'\mathbf{x}/(T_\star\sqrt{T_\star}) & \zeta'\zeta/(\sqrt{T_\star}\sqrt{T_\star}) \end{pmatrix}, \end{aligned} \quad (38)$$

where the  $T \rightarrow \infty$  and therefore the  $T_\star \rightarrow \infty$  limit of  $\mathbf{M}_{nT}^*$  is denoted by  $\mathbf{M}_n^*$ . In addition we define the  $d_x \times d_x$  matrices  $\mathbf{M}_{nTi}$  and  $\mathbf{M}_{nT}$ :

$$\begin{aligned} \mathbf{M}_{nTi} &= \left( \frac{1}{T_\star^2 n} \sum_{i=1}^n \sum_{t=1}^{T_\star} \mathbf{X}_{it,1:d_x} \mathbf{Z}'_{it,1:d_z} \right) \left( \frac{1}{T_\star^2 n} \sum_{i=1}^n \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} \right)^{-1} \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}'_{it,1:d_z} \mathbf{X}_{it,1:d_x} \\ \mathbf{M}_{nT} &= \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{nTi}. \end{aligned} \quad (39)$$

We denote their  $T \rightarrow \infty$  limits in distribution by  $\mathbf{M}_{ni}$  and  $\mathbf{M}_n$ , respectively.

**Remark 4.** In Remark 3 we already noted that the two-stage least squares estimator and the *DOLS* estimator are special cases of the dynamic two-stage least squares estimator. When we consider  $\mathbf{M}_{nTi}$  and assume that  $\mathbf{x}^* = \mathbf{y}^*$ , the product of the first two terms has to result in the identity matrix. In this case  $\mathbf{M}_{nTi} = \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}'_{it,1:d_z} \mathbf{X}_{it,1:d_x} = \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{X}'_{it,1:d_x} \mathbf{X}_{it,1:d_x}$ . This term exactly corresponds to the term  $\mathbf{M}_{nTi}$  in the *DOLS* paper of Mark and Sul (2003). The same argument holds with  $\mathbf{m}_{nTi}$ .

In the following steps we observe that  $\mathbf{M}_n$  is a submatrix of  $\mathbf{M}_n^*$ . To obtain the  $T \rightarrow \infty$  limit of  $\mathbf{M}_{nT}^*$ , we



are confronted with the terms

$$\frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \tilde{\mathbf{x}}_{it\lambda} \tilde{\mathbf{x}}_{jtu} \xrightarrow{d} \int \tilde{\mathbf{B}}_{vi\lambda} \tilde{\mathbf{B}}_{vju} , \quad \frac{1}{T_\star^\kappa} \sum_{t=1}^{T_\star} \tilde{\zeta}_{it\lambda} \tilde{\mathbf{x}}_{jtu} \xrightarrow{p} 0 , \quad \frac{1}{T_\star^\kappa} \sum_{t=1}^{T_\star} \tilde{u}_{it\lambda} \tilde{\mathbf{x}}_{jtu} \xrightarrow{p} 0 , \quad (40)$$

where  $\kappa \geq \frac{3}{2}$ . The terms of the form of  $\frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \tilde{\mathbf{x}}_{it} \tilde{\mathbf{x}}_{it} \xrightarrow{d} \int \tilde{\mathbf{B}}_{vi} \tilde{\mathbf{B}}_{vi}'$ .<sup>9</sup> In addition we meet terms of the structure

$$\frac{1}{\sqrt{T_\star} \sqrt{T_\star}} \sum_{t=1}^{T_\star} \tilde{\zeta}_{it} \tilde{\zeta}_{it} \xrightarrow{p} \begin{cases} \Gamma_{vv,ij}^\zeta & \text{for } j \in \mathbb{Z} \text{ and} \\ \mathbf{0}_{k \times k} & \text{else.} \end{cases} \quad (41)$$

$\frac{1}{\sqrt{T_\star} \sqrt{T_\star}} \sum_{t=1}^{T_\star} \tilde{\zeta}_{it} \tilde{\zeta}_{jt}$  converges to a matrix of zeros by the independence across  $i$  assumption (i.e. Assumption 2). For each fixed  $i = 1, \dots, n$ , the matrix  $\Gamma_{vv,ij}^\zeta$  contains the  $k \times k$  covariance matrices  $\Gamma_{vv,ij}$ , where  $j \in \mathbb{Z}$ .

Consider the now terms in (38). By the above arguments each of the three matrices converges to a block diagonal matrix. For the first matrix we obtain a non-zero block in the north-west of dimension  $d_x \times d_x$ , and a non-zero block consisting of  $\Gamma_{vv,ij}$ . The south-west and the north-east blocks are zero. With the second matrix we observe almost the same effect. The non-zero north-west block is of dimension  $d_z \times d_z$ , the south-east block is the same as the south-east block of the first matrix. The south-west and the north-east blocks are zero. The third matrix is the transpose of the first matrix. Therefore, the limit matrix  $\mathbf{M}_n^\star$  is block diagonal. From  $\mathbf{M}_n^\star$  we can extract the matrix  $\mathbf{M}_{ni}^\star$  focusing on the index  $i$ . The limit of the submatrix  $[\mathbf{M}_{ni}^\star]_{(1:k+1, 1:k+1)}$  is  $\mathbf{M}_{ni}$  while the limit of  $[\mathbf{M}_n^\star]_{(1:k+1, 1:k+1)}$  is  $\mathbf{M}_n$ .  $[\mathbf{M}_{ni}^\star]_{(k+2:k+1+(2p(T)+1)k \cdot n, k+2:k+1+(2p(T)+1)k \cdot n)}$  is a block diagonal matrix consisting of  $\Gamma_{vv,ij}^\zeta$ . The elements in the north-eastern and the south-western blocks of  $\mathbf{M}_{ni}^\star$  and  $\mathbf{M}_n^\star$  are zero. By this result in the

<sup>9</sup>The second and the third term converge to zero in probability. This also follows from Johansen (1995)[Chapter 13 & Appendix], Saikkonen (1991) and Davidson (1994). We already know that (e.g. Davidson (1994)[Theorem 30.13])

$$\frac{1}{T_\star} \sum_{t=1}^{T_\star} \tilde{\zeta}_{it\lambda} \tilde{\mathbf{x}}_{jtu} \xrightarrow{d} \int d\tilde{\mathbf{B}}_{vi\lambda} \tilde{\mathbf{B}}_{vju} \text{ and } \frac{1}{T_\star} \sum_{t=1}^{T_\star} \tilde{u}_{it\lambda} \tilde{\mathbf{x}}_{jtu} \xrightarrow{d} \sqrt{\Omega_{uu,i}} \int \tilde{\mathbf{B}}_{vju} d\mathcal{W}_{ui} .$$

A random variable convergent in distribution is bounded in probability, or  $\mathcal{O}_p(1)$  in Landau notation (see e.g. White (2001)[Lemma 4.5]). We can now consider  $\frac{1}{T_\star^{3/2}} \sum_{t=1}^{T_\star} \tilde{\zeta}_{it\lambda} \tilde{\mathbf{x}}_{jtu}$  as the product  $a \cdot b$ , where  $a = \frac{1}{\sqrt{T_\star}}$  and  $b = \frac{1}{T_\star} \sum_{t=1}^{T_\star} \tilde{\zeta}_{it\lambda} \tilde{\mathbf{x}}_{jtu}$ . Since  $a$  is converging to zero, it is  $o(1)$  and therefore also  $o_p(1)$ .  $b$  converges in distribution and therefore (its Euclidian norm) is  $\mathcal{O}_p(1)$ . We obtain convergence in probability to zero since the product  $o_p(1)\mathcal{O}_p(1)$  behaves like  $o_p(1)$ . Landau symbols are e.g. discussed in Poirier (1995)[page 196].

limit *only* the first  $d_x$  and  $d_z$  columns have an impact of the estimates of  $\gamma$ , while the remaining non-zero block affects the estimates of  $\delta$ . Next we consider  $\mathbf{X}'\mathbf{P}_H\mathbf{u}$ . Let us define the following terms:

**Definition 2.** Consider the  $1 + k + (2p(T) + 1)k \cdot n$  dimensional vector

$$\mathbf{m}_{nT}^* = \begin{pmatrix} \mathbf{y}^*\mathbf{x}^*/(T_\star^2 n) & \mathbf{y}^*\mathbf{x}/(T_\star^2 n) & \mathbf{y}^*\zeta/(T_\star\sqrt{T_\star}) \\ \mathbf{x}'\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}'\mathbf{x}/(T_\star^2 n) & \mathbf{x}'\zeta/(T_\star\sqrt{T_\star}) \\ \zeta'\mathbf{x}^*/(T_\star\sqrt{T_\star}) & \zeta'\mathbf{x}/(T_\star\sqrt{T_\star}) & \zeta'\zeta/(\sqrt{T_\star}\sqrt{T_\star}) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}^*\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}^*\mathbf{x}/(T_\star^2 n) & \mathbf{x}^*\zeta/(T_\star\sqrt{T_\star}) \\ \mathbf{x}'\mathbf{x}^*/(T_\star^2 n) & \mathbf{x}'\mathbf{x}/(T_\star^2 n) & \mathbf{x}'\zeta/(T_\star\sqrt{T_\star}) \\ \zeta'\mathbf{x}^*/(T_\star\sqrt{T_\star}) & \zeta'\mathbf{x}/(T_\star\sqrt{T_\star}) & \zeta'\zeta/(\sqrt{T_\star}\sqrt{T_\star}) \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{x}^*/(T_\star\sqrt{n}) \\ \mathbf{x}'/(T_\star\sqrt{n}) \\ \zeta'/\sqrt{T_\star} \end{pmatrix} \mathbf{u}. \quad (42)$$

The  $T \rightarrow \infty$  limit is denoted by  $\mathbf{m}_n^*$ . In addition we define

$$\mathbf{m}_{nTi} = \left( \frac{1}{T_\star^2 n} \sum_{i=1}^n \sum_{t=1}^{T_\star} \mathbf{X}_{it,1:d_x} \mathbf{Z}'_{it,1:d_z} \right) \left( \frac{1}{T_\star^2 n} \sum_{i=1}^n \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} \right)^{-1} \frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{u}_{it},$$

$$\mathbf{m}_{nT} = \frac{1}{\sqrt{n}} \mathbf{m}_{nTi}. \quad (43)$$

We denote their  $T \rightarrow \infty$  limits by  $\mathbf{m}_{ni}$  and  $\mathbf{m}_n$  respectively.

The first and the second matrix have already been considered with (38), where we have already observed that only the first  $d_x$  and  $d_z$  elements affect  $\gamma$  as  $T \rightarrow \infty$ . The product of these two matrices is multiplied with  $\sum_{t=1}^{T_\star} (\mathbf{Z}_{it} \cdot D_T) \mathbf{u}_{it}$ , where  $D_T = \left( \left( \frac{1}{T_\star} \cdot \mathbf{1}_{d_z} \right)', \left( \frac{1}{\sqrt{T_\star}} \cdot \mathbf{1}_{(2p(T)+1)k \cdot n} \right)' \right)'$  and  $\mathbf{1}_a$  is a vector of ones of dimension  $a$ . Last but not least Saikkonen (1991)[Theorem 4.1/ Lemma A.5] has shown that for the truncation error  $\mathbf{e}_{it}$  we observe that  $\|\sum_{t=1}^{T_\star} (\mathbf{Z}_{it} \cdot D_T) \mathbf{e}_{it}\|_2 = o_p(p^{1/2})$  such that  $\mathbf{u}_{it}$  converges in probability to  $\mathbf{u}_{it}$  as  $T \rightarrow \infty$ .  $\mathbf{u}_{it}$  is uncorrelated with the  $\tilde{\zeta}_{it} = \mathbf{Z}_{it,d_z+1:d_z+(2p(T)+1)k \cdot n}$ . By this we observe that only the first  $d_x$  components of  $\mathbf{X}_{it}$  and the first  $d_z$  components of  $\mathbf{Z}_{it}$  enter into the limit of the estimator  $\gamma$ . Therefore we have arrived at the *first result*: When  $T \rightarrow \infty$  the limit distribution of  $\gamma$  is given by the inverse of  $\mathbf{M}_n$  times  $\mathbf{m}_n$ . By the block diagonal structure obtained in the above paragraphs, elements of

$\mathbf{m}_{nT}^*$  and  $\mathbf{M}_{nT}^*$  outside  $1 : d_x$  and  $1 : d_x \times 1 : d_x$  do not affect the asymptotic distribution of  $\gamma$ . Hence  $\gamma$  and  $\delta$  are asymptotically independent. The submatrix  $[\mathbf{M}_{nT}^*]_{(k+2:k+1+(2p(T)+1)k:n, k+2:k+1+(2p(T)+1)k:n)}$  converges to a matrix consisting of  $\Gamma_{vv,ij}$ ,  $\gamma$  and  $\delta_i$  are independent for  $i = 1, \dots, n$ .

**Step 2:** Based on this asymptotic independence result we are permitted to focus on the matrix  $\mathbf{M}_{nT}$  and on the  $k + 1$  dimensional vector  $\mathbf{m}_{nT}$  to investigate the limit behavior of  $\gamma$ . In more details:

$$\begin{aligned} \mathbf{M}_{ZZ, nTi} &:= \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}_{it, 1:d_z} \mathbf{Z}'_{it, 1:d_z} \\ &= \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \left( \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{x}_{jt1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{x}_{j t q_\rho}, \tilde{\mathbf{x}}'_{it} \right) \left( \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{x}_{jt1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{x}_{j t q_\rho}, \tilde{\mathbf{x}}'_{it} \right) \end{aligned} \quad (44)$$

Using the results for the functional central limit we derive

$$\frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}_{it, 1:d_z} \mathbf{Z}'_{it, 1:d_z} \xrightarrow{d} \mathbf{M}_{ZZ, ni} \quad (45)$$

where

$$\begin{aligned} \mathbf{M}_{ZZ, ni, (1:q_\rho, 1:q_\rho)} &:= \\ &\begin{pmatrix} \int \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1} \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1}, \dots, \int \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1} \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vj q_\rho} \\ \vdots \\ \int \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vj q_\rho} \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1}, \dots, \int \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vj q_\rho} \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vj q_\rho} \end{pmatrix}, \\ \mathbf{M}_{ZZ, ni, (1:q_\rho, q_\rho+1:q_\rho+k)} &:= \begin{pmatrix} \int \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vi1}, \dots, \int \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vik} \\ \vdots \\ \int \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vj q_\rho} \tilde{\mathcal{B}}_{vi1}, \dots, \int \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vj q_\rho} \tilde{\mathcal{B}}_{vik} \end{pmatrix}, \\ \mathbf{M}_{ZZ, ni, (q_\rho+1:q_\rho+k, 1:q_\rho)} &:= \mathbf{M}'_{ZZ, ni, (1:q_\rho, q_\rho+1:q_\rho+k)} \\ \mathbf{M}_{ZZ, ni, (q_\rho+1:q_\rho+k, q_\rho+1:q_\rho+k)} &:= \int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}'_{vi}. \end{aligned}$$

Based on (45) we arrive at

$$\mathbf{M}_{ZZ, Ti} = \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{ZZ, nTi} \xrightarrow{d} \mathbf{M}_{ZZ, n} = \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{ZZ, ni}. \quad (46)$$

In a similar way we derive the limit of

$$\mathbf{X}_{it,1:d_x} \mathbf{Z}'_{it,1:d_z} = \left( \sum_{j=1}^n W_{ij} \tilde{y}_{jt}, \tilde{\mathbf{x}}'_{it} \right)' \left( \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{x}_{jt1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{x}_{jtq_\rho}, \tilde{\mathbf{x}}'_{it} \right). \quad (47)$$

Based on (17) we obtain  $y_{jt} = \sum_{l=1}^n K_{jl} (\beta' \mathbf{x}_{lt} + \delta_l \zeta_{lt} + \underline{u}_{lt})$ , where  $K_{jl} = [(\mathbf{I} - \rho \mathbf{W})]_{jl}$ . Then

$$\sum_{j=1}^n W_{ij} \tilde{y}_{jt} = \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} (\beta' \tilde{\mathbf{x}}_{lt} + \delta_l \tilde{\zeta}_{lt} + \tilde{\underline{u}}_{lt}). \quad (48)$$

By means of (47) and (48) we arrive at the  $k+1 \times k+q_\rho$  matrix

$$\begin{aligned} \mathbf{M}_{XZ,nTi} &= \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \left( \sum_{j=1}^n W_{ij} \tilde{y}_{jt}, \tilde{\mathbf{x}}'_{it} \right)' \left( \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathbf{x}}_{jt1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathbf{x}}_{jtq_\rho}, \tilde{\mathbf{x}}'_{it} \right) \\ &= \left( \begin{array}{c|c} \mathbf{M}_{XZ,nTi,(1:1,1:q_\rho)} & \mathbf{M}_{XZ,nTi,(1:1,q_\rho+1:k+q_\rho)} \\ \hline \mathbf{M}_{XZ,nTi,(2:k+1,1:q_\rho)} & \mathbf{M}_{XZ,nTi,(2:k+1,q_\rho+1:k+q_\rho)} \end{array} \right), \end{aligned} \quad (49)$$

where

$$\begin{aligned} \mathbf{M}_{XZ,nTi,(1:1,1:q_\rho)} &:= \\ \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} &\left( \begin{array}{c} \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} (\beta' \tilde{\mathbf{x}}_{lt} + \delta_l \tilde{\zeta}_{lt} + \tilde{\underline{u}}_{lt}) \cdot \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \tilde{x}_{\kappa t1} \\ \vdots \\ \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} (\beta' \tilde{\mathbf{x}}_{lt} + \delta_l \tilde{\zeta}_{lt} + \tilde{\underline{u}}_{lt}) \cdot \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \tilde{x}_{\kappa tq_\rho} \end{array} \right)', \\ \mathbf{M}_{XZ,nTi,(1:1,q_\rho+1:k+q_\rho)} &:= \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} (\beta' \tilde{\mathbf{x}}_{lt} + \delta_l \tilde{\zeta}_{lt} + \tilde{\underline{u}}_{lt}) \cdot \tilde{\mathbf{x}}'_{it}, \\ \mathbf{M}_{XZ,nTi,(2:k+1,1:q_\rho)} &:= \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \left( \tilde{\mathbf{x}}_{it} \cdot \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \tilde{x}_{\kappa t1}, \dots, \tilde{\mathbf{x}}_{it} \cdot \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \tilde{x}_{\kappa tq_\rho} \right), \\ \mathbf{M}_{XZ,nTi,(2:k+1,q_\rho+1:k+q_\rho)} &:= \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \tilde{\mathbf{x}}_{it} \tilde{\mathbf{x}}'_{it}. \end{aligned}$$

The  $T \rightarrow \infty$  limit of the matrix  $\mathbf{M}_{XZ,nTi}$  is given by:

$$\mathbf{M}_{XZ,ni} = \left( \begin{array}{c|c} \mathbf{M}_{XZ,ni,(1:1,1:q_\rho)} & \mathbf{M}_{XZ,ni,(1:1,q_\rho+1:k+q_\rho)} \\ \hline \mathbf{M}_{XZ,ni,(2:k+1,1:q_\rho)} & \mathbf{M}_{XZ,ni,(2:k+1,q_\rho+1:k+q_\rho)} \end{array} \right),$$

where

$$\mathbf{M}_{XZ,nTi,(1:1,1:q_\rho)} \xrightarrow{d} \left( \begin{array}{c} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} \left( \beta' \tilde{\mathcal{B}}_{vl} \tilde{\mathcal{B}}_{v\kappa 1} \right) \\ \vdots \\ \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} \left( \beta' \tilde{\mathcal{B}}_{vl} \tilde{\mathcal{B}}_{v\kappa q_\rho} \right) \end{array} \right)' = \mathbf{M}_{XZ,ni,(1:1,1:q_\rho)}, \quad (50)$$

$$\mathbf{M}_{XZ,nTi,(1:1,q_\rho+1+q_\rho+k)} \xrightarrow{d} \left( \begin{array}{c} \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} \left( \beta' \tilde{\mathcal{B}}_{vl} \tilde{\mathcal{B}}_{vi1} \right) \\ \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} \left( \beta' \tilde{\mathcal{B}}_{vl} \tilde{\mathcal{B}}_{vi2} \right) \\ \vdots \\ \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} \left( \beta' \tilde{\mathcal{B}}_{vl} \tilde{\mathcal{B}}_{vik} \right) \end{array} \right)' = M_{XZ,ni,(1:1,q_\rho+1+q_\rho+k)}, \quad (51)$$

$$\mathbf{M}_{XZ,nTi,(2+k+1,1:q_\rho)} \xrightarrow{d} \left( \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}_{v\kappa 1}, \dots, \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}_{v\kappa q_\rho} \right) = \mathbf{M}_{XZ,ni,(2:k+1,1:q_\rho)}, \quad (52)$$

$$\mathbf{M}_{XZ,nTi,(2+k+1,q_\rho+1:q_\rho+k)} \xrightarrow{d} \int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}_{vi}' = M_{XZ,ni,(2:k+1,q_\rho+1:q_\rho+k)}. \quad (53)$$

Note that  $\tilde{\mathcal{B}}_{vj}$  is a scalar while  $\tilde{\mathcal{B}}_{vi}$  a  $k$  dimensional vector. Summing up we arrive at

$$\mathbf{M}_{XZ,nT} = \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{XZ,nTi} \xrightarrow{d} \mathbf{M}_{XZ,n} = \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{XZ,ni}. \quad (54)$$

**Remark 5.** By Assumption 5 we have assumed that the matrix  $\mathbf{M}_{XZ,n}$  has rank  $k+1$ , while the matrix  $\mathbf{M}_{ZZ,n}$  has rank  $k+q_\rho \geq k+1$ . Therefore  $\mathbf{M}_{XZ,n} (\mathbf{M}_{ZZ,n})^{-1} \mathbf{M}_{ZX,n}$  has rank  $k+1$ . Lemma 1 shows that

this assumption is non-empty. If the conditions of Lemma 1 hold, then the matrices  $\mathbf{M}_{XZ,n}$  and  $\mathbf{M}_{ZZ,n}$  have rank,  $k + 1$  and  $k + q_\rho$ , respectively.

Next we derive  $\mathbf{m}_{ni}$  and  $\mathbf{m}_n$ . For the term  $\frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{X}_{it,2:d_x} \underline{u}_{it} = \frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{x}_{it,1:d_x} \underline{u}_{it}$  the  $T_\star \rightarrow \infty$  limit is already given by  $\sqrt{\Omega_{uu,i}} \int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui}$ . By using the functional central limit theorem and Saikonen (1991)[Theorem 4.1/ Lemma A.5] (e.g.  $\|\sum_{t=1}^{T_\star} (\mathbf{Z}_{it} \cdot D_T) \mathbf{e}_{it}\|_2 = o_p(p^{1/2})$ ) such that  $\underline{u}_{it} = u_{it}$  as  $T \rightarrow \infty$ ) the term  $\frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \underline{u}_{it}$  converges in distribution to

$$\mathbf{m}_{niZu} = \sqrt{\Omega_{uu,i}} \left( \sum_{j=1}^n W_{ij}^{\tau_1} \int \tilde{\mathcal{B}}_{vj1} d\mathcal{W}_{ui}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \int \tilde{\mathcal{B}}_{vjq_\rho} d\mathcal{W}_{ui}, \left( \int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui} \right)' \right)'. \quad (56)$$

To obtain the first term of  $\mathbf{m}_n$  we have to combine  $\mathbf{M}_{ZZ,n}$  provided by (46),  $\mathbf{M}_{XZ,ni}$  given by (55) and  $\mathbf{m}_{niZu}$ . Then the continuous mapping theorem yields

$$\mathbf{m}_{nT} \xrightarrow{d} \frac{1}{\sqrt{n}} \sum_{i=1}^n \sqrt{\Omega_{uu,i}} \left( \mathbf{M}_{XZ,n} \mathbf{M}_{ZZ,n}^{-1} \begin{pmatrix} \sum_{j=1}^n W_{ij}^{\tau_1} \int \tilde{\mathcal{B}}_{vj1} d\mathcal{W}_{ui} \\ \vdots \\ \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \int \tilde{\mathcal{B}}_{vjq_\rho} d\mathcal{W}_{ui} \\ \int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui} \end{pmatrix} \right)'. \quad (57)$$

$\mathbf{M}_{ZZ,n}$  is a  $k + q_\rho \times k + q_\rho$  matrix, while  $\mathbf{m}_{niZu}$  as well as

$$\left( \sum_{j=1}^n \int W_{ij}^{\tau_1} \tilde{\mathcal{B}}_{vj1} d\mathcal{W}_{ui}, \dots, \sum_{j=1}^n \int W_{ij}^{\tau_{q_\rho}} \tilde{\mathcal{B}}_{vjq_\rho} d\mathcal{W}_{ui}, \left( \int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui} \right)' \right)'$$

are vectors of dimension  $q_\rho + k$ . The elements 2 to  $k+1$  of  $\mathbf{m}_n$  are given by a sum of the  $k$  dimensional vectors  $\int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui}$ . Since the application of the projection operator  $\mathbf{P}_H$  on  $\mathbf{X}_{i,2:d_x}$  is  $\mathbf{X}_{i,2:d_x}$  (see equation (37)),

the rows  $2 : k + 1$  have to be equal to the limit of  $\frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{X}_{it,2:d_x} \mathbf{u}_{it} = \frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{x}_{it} \mathbf{u}_{it}$ . This yields

$$\mathbf{m}_{nT} \xrightarrow{d} \frac{1}{\sqrt{n}} \sum_{i=1}^n \sqrt{\Omega_{uu,i}} \left( \begin{array}{c} \left[ \mathbf{M}_{XZ,n} \mathbf{M}_{ZZ,n}^{-1} \begin{pmatrix} \sum_{j=1}^n W_{ij}^{T_1} \int \tilde{\mathcal{B}}_{vj1} d\mathcal{W}_{ui} \\ \vdots \\ \sum_{j=1}^n W_{ij}^{\tau_{q\rho}} \int \tilde{\mathcal{B}}_{vjq\rho} d\mathcal{W}_{ui} \\ \int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui} \end{pmatrix} \right]_{1,1} \\ \hline \int \tilde{\mathcal{B}}_{vi} d\mathcal{W}_{ui} \end{array} \right) = \mathbf{m}_n . \quad (58)$$

It remains to calculate the limit distribution of (38). By the asymptotic independence arguments for  $\gamma$  and  $\delta$ , we are allowed to restrict to  $\mathbf{X}'_{1:d_x} \mathbf{Z}_{1:d_z} (\mathbf{Z}'_{1:d_z} \mathbf{Z}_{1:d_z})^{-1} \mathbf{Z}'_{1:d_z} \mathbf{X}_{1:d_x}$  (weighted by  $1/T_\star$  and  $1/\sqrt{n}$ ). Using the above results and the continuous mapping theorem we get

$$\begin{aligned} \mathbf{M}_{ni} &= \mathbf{M}_{XZ,n} \mathbf{M}_{ZZ,n}^{-1} \mathbf{M}'_{XZ,ni}, \\ \mathbf{M}_n &= \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{ni} = \frac{1}{n} \mathbf{M}_{XZ,n} \mathbf{M}_{ZZ,n}^{-1} \mathbf{M}'_{XZ,n} . \end{aligned} \quad (59)$$

This yields the *second result*:  $(\gamma', \delta)'$  can be consistently estimated.  $\sqrt{n} T_\star (\hat{\gamma}_{D2SLS} - \gamma)$  converges in distribution to  $\mathbf{M}_n^{-1} \mathbf{m}_n$  as  $T \rightarrow \infty$ , where  $\mathbf{m}_n$  and  $\mathbf{M}_n$  are given by (58) and (59), respectively. With these estimates we can derive the residuals, which allow us to consistently estimate  $\Omega_{uu,i}$ .

**Step 3:** Finally we construct the Wald statistic  $S_{\gamma,n}$ . We follow Phillips and Hansen (1990), Johansen (1995) and Park and Phillips (1988) to derive the so called observed Wald-statistic  $S_{\gamma,nT}$  and its limit  $S_{\gamma,n}$ . Consider the  $s \times k + 1$  restriction matrix  $\mathbf{R}$ . Since S-ancillarity is implied by strong exogeneity as observed in our model, the ancillarity results presented in Johansen (1995) can be used. With  $\mathcal{B}_{vi}$  fixed for all  $i = 1, \dots, n$ : (i) the terms  $\mathbf{M}_{ni}$  and  $\mathbf{M}_n$  are constant matrices; (ii)  $\mathbf{m}_n$  is a mixed Gaussian vector

with mean zero and variance  $\mathbf{V}_n$  where<sup>10</sup>

$$\mathbf{V}_n = \frac{1}{n} \sum_{i=1}^n V_{ni} \text{ where } \mathbf{V}_{ni} = \Omega_{uu,i} \cdot \tilde{\Upsilon}_{ni} . \quad (60)$$

$$\begin{aligned} \tilde{\Upsilon}_{ni} &= \int \left[ \left( \begin{array}{c} [\mathbf{M}_{XZ,n} \mathbf{M}_{ZZ,n}^{-1} \mathbf{m}_{Z,n}]_{(1,1)} \\ \tilde{\mathbf{B}}_{vi} \end{array} \right) \left( \begin{array}{c} [\mathbf{M}_{XZ,n} \mathbf{M}_{ZZ,n}^{-1} \mathbf{m}_{Z,n}]_{(1,1)} \\ \tilde{\mathbf{B}}_{vi} \end{array} \right)' \right] , \\ \mathbf{m}_{Z,n} &= \left( \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathbf{B}}_{vj1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q\rho}} \tilde{\mathbf{B}}_{vjq\rho}, \tilde{\mathbf{B}}'_{vi} \right)' . \end{aligned} \quad (61)$$

Then the asymptotic covariance matrix of  $\sqrt{n}T_\star (\gamma_{D2SLS} - \gamma)$  becomes

$$\mathbf{D}_n = \mathbf{M}_n^{-1} \mathbf{V}_n \mathbf{M}_n^{-1} . \quad (62)$$

An estimate  $\mathbf{V}_{nT}$  of  $\mathbf{V}_n$  is derived by means of

$$\begin{aligned} \mathbf{V}_{nT} &= \frac{1}{n} \sum_{i=1}^n \hat{\Omega}_{uu,i} \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \Upsilon_{nTi} \Upsilon'_{nTi} \\ \Upsilon_{nTi} &= \left( \begin{array}{c} \left[ \left( \sum_{l=1}^n \sum_{t=1}^{T_\star} \mathbf{X}_{lt,1:d_x} \mathbf{Z}'_{lt,1:d_z} \right) \left( \sum_{j=1}^n \sum_{t=1}^{T_\star} \mathbf{Z}_{jt,1:d_z} \mathbf{Z}'_{jt,1:d_z} \right)^{-1} \mathbf{Z}_{it,1:d_z} \right]_{(1,1)} \\ \mathbf{x}_{it} \end{array} \right) . \end{aligned} \quad (63)$$

Combining (63) and  $\mathbf{M}_{nT}$ , which is an estimate of  $\mathbf{M}_n$ , we arrive at an estimate of the covariance matrix

$$\mathbf{D}_{nT} = \mathbf{M}_{nT}^{-1} \mathbf{V}_{nT} \mathbf{M}_{nT}^{-1} . \quad (64)$$

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<sup>10</sup>Note that as with  $\mathbf{m}_{nT}$  the term  $\mathbf{M}_{XZ,Tn} \mathbf{M}_{ZZ,Tn}^{-1} \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \left( \sum_{j=1}^n W_{ij}^{\tau_1} Z_{it,1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q\rho}} Z_{it,q\rho}, \tilde{\mathbf{x}}'_{it} \right)'$  is equal to  $\left( \left[ \mathbf{M}_{XZ,Tn} \mathbf{M}_{ZZ,Tn}^{-1} \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \left( \sum_{j=1}^n W_{ij}^{\tau_1} Z_{it,1}, \dots, \sum_{j=1}^n W_{ij}^{\tau_{q\rho}} Z_{it,q\rho}, \tilde{\mathbf{x}}'_{it} \right)' \right]_{1,1}, \tilde{\mathbf{x}}'_{it} \right)'$  by the fact the projection  $\mathbf{P}_H$  applied to  $\tilde{\mathbf{x}}_{it}$  is  $\tilde{\mathbf{x}}_{it}$ .



Equipped with these terms we obtain

$$\begin{aligned}
S_{\gamma, nT} &= (\sqrt{n}T_\star \mathbf{R} (\hat{\gamma}_{D2SLS} - \gamma))' (\mathbf{R} \mathbf{D}_{nT} \mathbf{R}')^{-1} (\sqrt{n}T_\star \mathbf{R} (\hat{\gamma}_{D2SLS} - \gamma)) , \\
S_{\gamma, nT} &\xrightarrow{d} S_{\gamma, n} = (\sqrt{n}T_\star \mathbf{R} (\hat{\gamma}_{D2SLS} - \gamma))' (\mathbf{R} \mathbf{D}_n \mathbf{R}')^{-1} (\sqrt{n}T_\star \mathbf{R} (\hat{\gamma}_{D2SLS} - \gamma)) .
\end{aligned} \tag{65}$$

Under the null hypothesis the Wald statistic  $S_{\gamma, nT}$  follows a  $\chi^2$  distribution with  $s$  degrees of freedom. This yields the *third result*:  $S_{\gamma, nT} \xrightarrow{d} S_{\gamma, n}$ ;  $\mathbf{D}_{nT}$  provides us with an estimate of the asymptotic covariance of the estimator  $\mathbf{D}_n$ .

**Remark 6** (Identification). From Deistler and Seifert (1978)[Theorem 4] it follows that if there exists a consistent estimator for the internal characteristics  $C_I$  of the model, the model is identifiable (i.e. an *identifying function* exists). In our setup the internal characteristics can be described by means of the parameters  $\rho$  and  $\beta$  and the covariances of the noise. Based on our Theorem 1 we have obtained a (super)consistent estimator for the parameters  $\rho, \beta$ . The covariance matrices can be estimated consistently due to Jansson (2002). The reasons why  $\hat{\gamma}_{D2SLS}$  becomes consistent are Assumption 5 on the instruments (Appendix B shows that this assumption is non-empty, there also Assumption 1 becomes important; in addition Assumption 3 is important to obtain the corresponding ranks of in limits required in Assumption 5) and Assumption 4 such that the truncation error goes to zero.

## B Instruments and the Rank Condition

Consider the instruments  $\tilde{x}_{itv}^* = \sum_{j=1}^n W_{ij}^{\tau_v} \tilde{x}_{jtv}$ ,  $v = 1, \dots, q_\rho$ . In the following we show that the requirement of valid instruments (see also Assumption 5) can be fulfilled under fairly mild restrictions. It is important to note that this implies that our "high level" Assumption 5 is non-empty. For the  $\mathbf{W}$  used in the applied part we use the following lemma:

**Lemma 1.** Given the model assumptions of Section 2. Suppose that  $\tilde{x}_{itv}^* = \sum_{j=1}^n W_{ij}^{\tau_v} \tilde{x}_{jtv}$ ,  $v = 1, \dots, q_\rho$  and  $\tau_v = 1$  and  $W_{ij} \neq 0$  for at least one  $j$ ,  $j \neq i$ , for each row  $i$ . Then the  $q_\rho + k \times q_\rho + k$  matrices  $\mathbf{M}_{ZZ,ni}$  and  $\mathbf{M}_{ZZ,n}$  have rank  $q_\rho + k$  almost surely. Additionally with  $\beta \neq \mathbf{0}_{k \times 1}$  the rank of the  $k + 1 \times q_\rho + k$  matrices  $\mathbf{M}_{XZ,ni}$  and  $\mathbf{M}_{XZ,n}$  is  $k + 1$  almost surely.

*Proof.* We consider the vectors

$$\begin{aligned} \mathbf{Z}_{it,1:d_z} &= \begin{pmatrix} \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{x}_{jt1} \\ \vdots \\ \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{x}_{jtg_\rho} \\ \tilde{x}_{it1} \\ \vdots \\ \tilde{x}_{itk} \end{pmatrix} \text{ and} \\ \mathbf{X}_{it,1:d_x} &= \begin{pmatrix} \sum_{j=1}^n W_{ij} \tilde{y}_{jt} \\ \tilde{x}_{it1} \\ \vdots \\ \tilde{x}_{itk} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \sum_{l=1}^n W_{ij} K_{jl} (\beta' \tilde{\mathbf{x}}_{lt} + \delta'_i \zeta_{lt} + \tilde{u}_{lt}) \\ \tilde{x}_{it1} \\ \vdots \\ \tilde{x}_{itk} \end{pmatrix} \end{aligned} \quad (66)$$

of dimension  $q_\rho + k$  and  $1 + k$ , respectively.  $\sum_{j=1}^n W_{ij} \tilde{y}_{jt}$  follows from (48), where  $K_{jl} = [(\mathbf{I} - \rho \mathbf{W})]_{jl}$ . In the following we calculate the limits of  $\frac{1}{T^*} \sum_{t=1}^{T^*} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z}$ ,  $\frac{1}{n} \sum_{i=1}^n \frac{1}{T^*} \sum_{t=1}^{T^*} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z}$ ,  $\frac{1}{T^*} \sum_{t=1}^{T^*} \mathbf{X}_{it,1:d_x} \mathbf{Z}'_{it,1:d_z}$  and  $\frac{1}{T^*} \sum_{i=1}^n \sum_{t=1}^{T^*} \mathbf{X}_{it,1:d_x} \mathbf{Z}'_{it,1:d_z}$ . Let us start with  $\sum_{t=1}^{T^*} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z}$  where we get

$$\begin{aligned}
& \sum_{t=1}^{T_*} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} = \\
& \left( \begin{array}{c|c}
\sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \sum_{l=1}^n W_{il}^{T_1} \tilde{x}_{lt1} & \dots & \sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \sum_{l=1}^n W_{il}^{T_{q\rho}} \tilde{x}_{ltq\rho} \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \sum_{l=1}^n W_{il}^{T_1} \tilde{x}_{lt1} & \dots & \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \sum_{l=1}^n W_{il}^{T_{q\rho}} \tilde{x}_{ltq\rho} \\
\tilde{x}_{it1} \sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} & \dots & \tilde{x}_{it1} \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \\
\vdots & \ddots & \vdots \\
\tilde{x}_{itk} \sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} & \dots & \tilde{x}_{itk} \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho}
\end{array} \right) \cdot \\
& \left( \begin{array}{c}
\sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \tilde{x}_{it1} & \dots & \sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \tilde{x}_{itk} \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{it1} & \dots & \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{itk} \\
\tilde{x}_{it1} \tilde{x}_{it1} & \dots & \tilde{x}_{it1} \tilde{x}_{itk} \\
\vdots & \ddots & \vdots \\
\tilde{x}_{itk} \tilde{x}_{it1} & \dots & \tilde{x}_{itk} \tilde{x}_{itk}
\end{array} \right) .
\end{aligned} \tag{67}$$

Since  $\sum_{j=1}^n W_{ij}^{T_v} \sum_{l=1}^n W_{il}^{T_w} \tilde{x}_{jtv} \tilde{x}_{ltw} = \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{T_v} W_{il}^{T_w} \tilde{x}_{jtv} \tilde{x}_{ltw}$  the matrix (67) can be written as:

$$\begin{aligned}
& \sum_{t=1}^{T_*} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} = \\
& \left( \begin{array}{c|c}
\sum_{l=1}^n \sum_{l=1}^n W_{ij}^{T_1} W_{il}^{T_1} \tilde{x}_{jt1} \tilde{x}_{lt1} & \dots & \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{T_1} W_{il}^{T_{q\rho}} \tilde{x}_{jt1} \tilde{x}_{ltq\rho} \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^n \sum_{l=1}^n W_{ij}^{T_{q\rho}} W_{il}^{T_1} \tilde{x}_{jtq\rho} \tilde{x}_{lt1} & \dots & \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{T_{q\rho}} W_{il}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{ltq\rho} \\
\sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \tilde{x}_{it1} & \dots & \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{it1} \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \tilde{x}_{itk} & \dots & \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{itk}
\end{array} \right) \cdot \\
& \left( \begin{array}{c}
\sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \tilde{x}_{it1} & \dots & \sum_{j=1}^n W_{ij}^{T_1} \tilde{x}_{jt1} \tilde{x}_{itk} \\
\vdots & \ddots & \vdots \\
\sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{it1} & \dots & \sum_{j=1}^n W_{ij}^{T_{q\rho}} \tilde{x}_{jtq\rho} \tilde{x}_{itk} \\
\tilde{x}_{it1} \tilde{x}_{it1} & \dots & \tilde{x}_{it1} \tilde{x}_{itk} \\
\vdots & \ddots & \vdots \\
\tilde{x}_{itk} \tilde{x}_{it1} & \dots & \tilde{x}_{itk} \tilde{x}_{itk}
\end{array} \right) .
\end{aligned} \tag{68}$$

(68) is a symmetric matrix. Now we set  $\tau_v = 1$  and take the limit of "(68) divided by  $T_\star^2$ ". For  $\tilde{x}_{itv}^\star = \sum_{j=1}^n W_{ij}^{\tau_v} \tilde{x}_{jtv}$ ,  $v = 1, \dots, q_\rho$  and  $\tau_v = 1$  we observe that  $q_\rho \leq k$ . By the functional central limit theorem we obtain

$$\mathbf{M}_{ZZ,ni} = \lim_{t \rightarrow \infty} \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} = \tag{69}$$

$$\begin{pmatrix} \int \sum_{j=1}^n \sum_{l=1}^n W_{ij} W_{il} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vl1} & \cdots & \int \sum_{j=1}^n \sum_{l=1}^n W_{ij} W_{il} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vlq_\rho} & \left| \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vi1} & \cdots & \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vik} \right. \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int \sum_{j=1}^n \sum_{l=1}^n W_{ij} W_{il} \tilde{\mathcal{B}}_{vjq_\rho} \tilde{\mathcal{B}}_{vl1} & \cdots & \int \sum_{j=1}^n \sum_{l=1}^n W_{ij} W_{il} \tilde{\mathcal{B}}_{vjq_\rho} \tilde{\mathcal{B}}_{vlq_\rho} & \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vjq_\rho} \tilde{\mathcal{B}}_{vi1} & \cdots & \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vjq_\rho} \tilde{\mathcal{B}}_{vik} \\ \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vi1} & \cdots & \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vjq_\rho} \tilde{\mathcal{B}}_{vi1} & \int \tilde{\mathcal{B}}_{vi1} \tilde{\mathcal{B}}_{vi1} & \cdots & \int \tilde{\mathcal{B}}_{vi1} \tilde{\mathcal{B}}_{vik} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vj1} \tilde{\mathcal{B}}_{vik} & \cdots & \int \sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vjq_\rho} \tilde{\mathcal{B}}_{vik} & \int \tilde{\mathcal{B}}_{vik} \tilde{\mathcal{B}}_{vi1} & \cdots & \int \tilde{\mathcal{B}}_{vik} \tilde{\mathcal{B}}_{vik} \end{pmatrix}.$$

When we consider (68) we observe that for any fixed  $t$  the row  $v$  is a linear combination of  $\mathbf{Z}'_{it,1:d_z}$  with the element  $\left[ \mathbf{Z}_{it,1:d_z} \right]_v$ . This also translates to the limit (69), where we observe that in each row each element includes a term arising from  $\left[ \mathbf{Z}_{it,1:d_z} \right]_v$ . E.g.  $\sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vj1}$  for the first row,  $\sum_{j=1}^n W_{ij} \tilde{\mathcal{B}}_{vjq_\rho}$  for row  $q_\rho$ ,  $\tilde{\mathcal{B}}_{vi1}$  for row  $q_\rho + 1, \dots$  and  $\tilde{\mathcal{B}}_{vik}$  for row  $d_z = q_\rho + k$ . The important ingredients to observe that the matrix  $\mathbf{M}_{ZZ,ni}$  is of full rank are (i) Assumption 3 (based on Phillips (2006)) to consider terms where  $i = j$ , (ii) Phillips and Hansen (1990)[Lemma A.3] to consider terms where  $i \neq j$  and (iii) the assumption that  $W_{ij} \neq 0$ . By  $W_{ij} \neq 0$  the terms in the first  $q_\rho$  rows and the first  $q_\rho$  columns of (69) are non-zero almost surely. Note that by Assumption 3 the matrix  $\int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}'_{vi}$  is positive definite and of full rank  $k$ . By Phillips and Hansen (1990)[Lemma A.3] the  $k \times k$  matrix  $\int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}'_{vj}$  is also positive definite and of rank  $k$  (almost surely).<sup>11</sup> By the assumption (in Lemma 1) that  $\mathbf{W}_{i,1:n}$  is not equal to a row vector of zeros, we meet a sum of matrices  $\int \tilde{\mathcal{B}}_{vj} \tilde{\mathcal{B}}'_{vl}$  weighted by  $W_{ij} W_{il}$  in the north-western part of the matrix  $\mathbf{M}_{ZZ,ni}$  in (69).  $W_{ij} \neq 0$  and  $W_{il} \neq 0$  for some  $j, l \neq i$  by Assumption 1; note that  $W_{ij} W_{il} \neq 0$  for  $j = l$ . For each pair  $j$  and  $l$  the term  $\int \tilde{\mathcal{B}}_{vj,1:q_\rho} \tilde{\mathcal{B}}'_{vl,1:q_\rho}$  has rank  $q_\rho$  by Assumption 3 or Phillips and Hansen (1990)[Lemma A.3]. Additionally, these terms are positive definite. Therefore, the term  $\int \sum_{j=1}^n \sum_{l=1}^n W_{ij} W_{il} \tilde{\mathcal{B}}_{vj,1:q_\rho} \tilde{\mathcal{B}}'_{vl,1:q_\rho}$  has rank  $q_\rho$  (see also Horn and Johnson (1985)[Obs. 7.1.3, page 398]). The south-eastern part has rank  $k$  by Assumption 3. The south-western part and the north-eastern part mix  $\tilde{\mathcal{B}}_{vl,1:q_\rho}$  with  $\tilde{\mathcal{B}}'_{vi}$ . Since  $q_\rho \leq k$  the ranks of these matrices are  $\leq k$ . Since the rank of the  $k + q_\rho \times q_\rho$  submatrix  $[\mathbf{M}_{ZZ,ni}]_{1:k+q_\rho, 1:q_\rho}$  is  $\leq k$ , there exist some scalars  $\lambda_{\ell 1}, \dots, \lambda_{\ell k}$  to express the rows  $\ell$ ,  $\ell = 1, \dots, q_\rho$ , by means of a linear combination

<sup>11</sup>Given the notation in Phillips and Hansen (1990)[Lemma A.3]. Our  $\mathbf{v}_i$  corresponds to  $\Delta x_2$ , while  $\mathbf{v}_j$  corresponds to  $\Delta x_3$ . If  $i = j$  in  $\int \tilde{\mathcal{B}}_{vi} \tilde{\mathcal{B}}'_{vj}$ , then the full rank process assumption is used, with  $i \neq j$  the Lemma A.3 can be applied.

of the other  $k$  rows. I.e. in the north-western part of the matrix  $\mathbf{M}_{ZZ,ni}$  we meet mixtures of components  $\tilde{\mathcal{B}}_{vj}$  with components of  $\tilde{\mathcal{B}}_{vl}$ ,  $l = 1, \dots, n$  where  $l \neq i$  ( $W_{ii} = 0$  by Assumption 1 while some  $W_{il} \neq 0$  as assumed above). In the north-eastern part of the matrix  $\mathbf{M}_{ZZ,ni}$  we meet mixtures of the components  $\tilde{\mathcal{B}}_{vj}$  with the components of  $\tilde{\mathcal{B}}_{vi}$ .  $\tilde{\mathcal{B}}_{vi} \neq \tilde{\mathcal{B}}_{vl}$  (almost surely) by the properties of Brownian motion. Therefore the corresponding rows of the north-eastern submatrix cannot be reconstructed by the mixture weights  $\lambda_{\ell_1}, \dots, \lambda_{\ell_k}$ . Note that in each of the rows  $1, \dots, q_\rho$  we meet different mixtures, from  $\tilde{\mathcal{B}}_{vj_1}$  in the first row to  $\tilde{\mathcal{B}}_{vj_{q_\rho}}$  in the  $q_\rho$ th row. That is to say we cannot rebuild rows  $\ell$ ,  $\ell \in \{1, \dots, q_\rho\}$  from the other rows. Since the matrix is symmetric this also holds for the columns. Summing up, the rank of the  $k + q_\rho \times k + q_\rho$  matrix  $\mathbf{M}_{ZZ,ni}$  is  $k + q_\rho$  (almost surely).

The limit of  $\frac{1}{n} \sum_{i=1}^n \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z}$  still has rank  $q_\rho + k$ . In the same way as for some  $\mathbf{M}_{ZZ,ni}$ , we cannot reconstruct the columns/rows from the other columns/rows when the sum  $\frac{1}{n} \cdot \sum_{i=1}^n \mathbf{M}_{ZZ,ni}$  is considered.

**Remark 7.** Note that  $\mathbf{M}_{ZZ,nTi}$ ,  $\mathbf{M}_{ZZ,nT}$ ,  $\mathbf{M}_{ZZ,ni}$  are  $\mathbf{M}_{ZZ,n}$  symmetric matrices.

In the next step we investigate  $\mathbf{M}_{XZ,nTi}$ . Hence we consider

$$\sum_{t=1}^{T_\star} \mathbf{x}_{it,1:d_z} \mathbf{z}'_{it,1:d_z} = \tag{70}$$

$$\sum_{t=1}^{T_\star} \left( \begin{array}{ccc|ccc} \sum_{j=1}^n W_{ij} \tilde{y}_{jt} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \tilde{x}_{\kappa t 1} & \cdots & \sum_{j=1}^n W_{ij} \tilde{y}_{jt} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \tilde{x}_{\kappa t q_\rho} & \sum_{j=1}^n W_{ij} \tilde{y}_{jt} \tilde{x}_{it 1} & \cdots & \sum_{j=1}^n W_{ij} \tilde{y}_{jt} \tilde{x}_{it k} \\ \tilde{x}_{it 1} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \tilde{x}_{\kappa t 1} & \cdots & \tilde{x}_{it 1} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \tilde{x}_{\kappa t q_\rho} & \tilde{x}_{it 1} \tilde{x}_{it 1} & \cdots & \tilde{x}_{it 1} \tilde{x}_{it k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{it k} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_1} \tilde{x}_{\kappa t 1} & \cdots & \tilde{x}_{it k} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_{q_\rho}} \tilde{x}_{\kappa t q_\rho} & \tilde{x}_{it k} \tilde{x}_{it 1} & \cdots & \tilde{x}_{it k} \tilde{x}_{it k} \end{array} \right).$$

Since  $\sum_{j=1}^n W_{ij} \tilde{y}_{jt} \sum_{\kappa=1}^n W_{i\kappa}^{\tau_w} \tilde{x}_{\kappa t w} = \sum_{j=1}^n \sum_{\kappa=1}^n W_{ij} W_{i\kappa}^{\tau_w} \tilde{y}_{jt} \tilde{x}_{\kappa t w}$  the matrix (70) can be written as:



Since  $W_{i,1:n} \neq \mathbf{0}_{1 \times n}$  and  $\beta \neq \mathbf{0}_{k \times 1}$  (which is assumed in Lemma 1) the north-western term in (72) is not equal to zero, this also follows for  $[\mathbf{M}_{XZ,ni}]_{1,q_\rho+1:k+q_\rho}$  and  $[\mathbf{M}_{XZ,ni}]_{2:2+k,1:q_\rho}$ . The rank of these matrices has to be  $\leq k$ . I.e. we can write the first row of the submatrix  $[\mathbf{M}_{XZ,ni}]_{1,q_\rho+1:k+q_\rho}$  as superposition of the remaining  $k$  rows of this submatrix. Due to the fact that  $W_{ij} \neq 0$  for some  $j$ , we observe that the first row of the matrix  $\mathbf{M}_{XZ,ni}$  cannot be expressed by the other rows (we also meet the index  $i$  in the remaining submatrix). Therefore, the matrix (72) has rank  $k + 1$  almost surely.

Last but not least by considering the limit of the sum  $\frac{1}{n} \sum_{i=1}^n \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{X}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z}$  we obtain  $\mathbf{M}_{XZ,n} = \frac{1}{n} \sum_{i=1}^n \mathbf{M}_{XZ,ni}$ . By the same reasoning as with  $\mathbf{M}_{XZ,ni}$  we observe that this sum of matrices has rank  $k + 1$ .  $\square$

**Remark 8.** Note that  $\mathbf{M}_{ZX,nTi} = \mathbf{M}'_{XZ,nTi}$ ,  $\mathbf{M}_{ZX,ni} = \mathbf{M}'_{XZ,ni}$ ,  $\mathbf{M}_{ZX,nT} = \mathbf{M}'_{XZ,nT}$  and  $\mathbf{M}_{ZX,n} = \mathbf{M}'_{XZ,n}$ .

If necessary the conditions of Lemma 1 can be replaced by less stringent assumptions. To keep the notation simple and by the fact that  $\tau_v = 1$  in the applied part, we presented first the Lemma with  $\tau_v = 1$ . One way to extend Lemma 1 is as follows:

**Lemma 2.** Given the model assumptions of Section 2 and the instruments  $\tilde{x}_{itv}^* = \sum_{j=1}^n W_{ij}^{\tau_v} \tilde{x}_{jtv}$ ,  $v = 1, \dots, q_\rho$  and  $q_\rho \leq k$ . Suppose that  $W_{ij}^{\tau_v} \neq 0$  for at least one  $j$  where  $j \neq i$ . Then the  $q_\rho + k \times q_\rho + k$  matrix  $\mathbf{M}_{ZZ,n}$  has rank  $q_\rho + k$  almost surely. Additionally with  $\beta \neq \mathbf{0}_{k \times 1}$  the rank of the  $k + 1 \times q_\rho + k$  matrix  $\mathbf{M}_{XZ,n}$  is  $k + 1$  almost surely.

*Proof.* By taking the sum over the components  $i = 1, \dots, n$  of the limits of "(68) divided by  $T_\star^2 n$ " we get  $\mathbf{M}_{ZZ,n}$ .  $\mathbf{M}_{ZZ,n}$  is equal to

$$\mathbf{M}_{ZZ,n} = \frac{1}{n} \sum_{i=1}^n \lim_{t \rightarrow \infty} \frac{1}{T_\star} \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} = \left( \begin{array}{c|c} [\mathbf{M}_{ZZ,ni}]_{1:q_\rho,1:q_\rho} & [\mathbf{M}_{ZZ,ni}]_{1:q_\rho,q_\rho+1:k+q_\rho} \\ \hline [\mathbf{M}_{ZZ,ni}]_{q_\rho+1:k+q_\rho,1:q_\rho} & [\mathbf{M}_{ZZ,ni}]_{q_\rho+1:k+q_\rho,q_\rho+1:k+q_\rho} \end{array} \right)$$

where

$$\begin{aligned} & [\mathbf{M}_{ZZ,ni}]_{1:q_\rho,1:q_\rho} = \\ & \frac{1}{n} \cdot \left( \begin{array}{ccc} \int \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{\tau_1} W_{il}^{\tau_1} \tilde{\mathbf{B}}_{vj1} \tilde{\mathbf{B}}_{vl1} & \cdots & \int \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{\tau_1} W_{il}^{\tau_{q_\rho}} \tilde{\mathbf{B}}_{vj1} \tilde{\mathbf{B}}_{vlq_\rho} \\ \vdots & \ddots & \vdots \\ \int \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{\tau_{q_\rho}} W_{il}^{\tau_1} \tilde{\mathbf{B}}_{vjq_\rho} \tilde{\mathbf{B}}_{vl1} & \cdots & \int \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n W_{ij}^{\tau_{q_\rho}} W_{il}^{\tau_{q_\rho}} \tilde{\mathbf{B}}_{vjq_\rho} \tilde{\mathbf{B}}_{vlq_\rho} \end{array} \right), \\ & [\mathbf{M}_{ZZ,ni}]_{q_\rho+1:k+q_\rho,1:q_\rho} = \frac{1}{n} \cdot \left( \begin{array}{ccc} \int \sum_{i=1}^n \tilde{\mathbf{B}}_{vi1} \sum_{l=1}^n W_{il}^{\tau_1} \tilde{\mathbf{B}}_{vl1} & \cdots & \int \sum_{i=1}^n \tilde{\mathbf{B}}_{vi1} \sum_{l=1}^n W_{il}^{\tau_{q_\rho}} \tilde{\mathbf{B}}_{vlq_\rho} \\ \vdots & \ddots & \vdots \\ \int \sum_{i=1}^n \tilde{\mathbf{B}}_{vik} \sum_{l=1}^n W_{il}^{\tau_1} \tilde{\mathbf{B}}_{vl1} & \cdots & \int \sum_{i=1}^n \tilde{\mathbf{B}}_{vik} \sum_{l=1}^n W_{il}^{\tau_{q_\rho}} \tilde{\mathbf{B}}_{vlq_\rho} \end{array} \right), \\ & [\mathbf{M}_{ZZ,ni}]_{1:q_\rho,q_\rho+1:k+q_\rho} = \frac{1}{n} \cdot \left( \begin{array}{ccc} \int \sum_{i=1}^n \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathbf{B}}_{vj1} \tilde{\mathbf{B}}_{vi1} & \cdots & \int \sum_{i=1}^n \sum_{j=1}^n W_{ij}^{\tau_1} \tilde{\mathbf{B}}_{vj1} \tilde{\mathbf{B}}_{vik} \\ \vdots & \ddots & \vdots \\ \int \sum_{i=1}^n \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathbf{B}}_{vjq_\rho} \tilde{\mathbf{B}}_{vi1} & \cdots & \int \sum_{i=1}^n \sum_{j=1}^n W_{ij}^{\tau_{q_\rho}} \tilde{\mathbf{B}}_{vjq_\rho} \tilde{\mathbf{B}}_{vik} \end{array} \right), \\ & [\mathbf{M}_{ZZ,ni}]_{q_\rho+1:k+q_\rho,q_\rho+1:k+q_\rho} = \frac{1}{n} \cdot \int \sum_{i=1}^n \tilde{\mathbf{B}}_{vi} \tilde{\mathbf{B}}_{vi}' . \end{aligned} \quad (73)$$

First, the south-eastern part  $[\mathbf{M}_{ZZ,ni}]_{q_\rho+1:k+q_\rho,q_\rho+1:k+q_\rho}$  is a sum of positive definite matrices of rank  $k$  (see Assumption 3), by Assumption 2 these matrices are also independent. Therefore this block is of rank  $k$  almost surely. Second, we consider the submatrix  $[\mathbf{M}_{ZZ,ni}]_{1:q_\rho+k,1:q_\rho}$  where we meet the terms  $\tilde{\mathbf{B}}_{vi} \tilde{\mathbf{B}}_{vj}'$ , possibly with  $i = j$ . By the assumption of this lemma at least one term  $W_{ij} \neq 0$  with  $i \neq j$ . Therefore  $[\mathbf{M}_{ZZ,ni}]_{1:q_\rho+k,1:q_\rho} \neq \mathbf{0}_{q_\rho+k,q_\rho}$ . Since  $q_\rho \leq k$  the rank of the submatrix  $[\mathbf{M}_{ZZ,ni}]_{1:q_\rho+k,1:q_\rho}$  is  $\leq \min\{k, q_\rho\}$  and therefore smaller or equal to  $k$ . I.e. we can express row  $\ell \in \{1, \dots, q_\rho\}$  by some linear combination of the rows  $q_\rho + 1, \dots, k + q_\rho$  with weights  $\lambda_{\ell 1}, \dots, \lambda_{\ell k}$ . Now we take these weights and the submatrix  $[\mathbf{M}_{ZZ,ni}]_{1:q_\rho+k,q_\rho+1:k+q_\rho}$ . Let us try to express row  $\ell$  by means of a linear combination of the weights  $\lambda_{\ell 1}, \dots, \lambda_{\ell k}$  and the rows  $q_\rho + 1$  to  $q_\rho + k$  of this submatrix. This would require to express terms of the structure  $\tilde{\mathbf{B}}_{vl} \tilde{\mathbf{B}}_{vj}'$ , where  $j, l \neq i$  for at least one summand, by means of the terms  $\tilde{\mathbf{B}}_{vi} \tilde{\mathbf{B}}_{vi}'$  which is not possible by the properties of the Brownian motion. In other words since  $\tilde{\mathbf{B}}_{vi} \neq \tilde{\mathbf{B}}_{vj}$  (almost surely), this cannot work for row  $\ell$ . In addition we observe that we mix with  $\tilde{\mathbf{B}}_{vj1}$  in the first row, with  $\tilde{\mathbf{B}}_{vj2}$  in the



second row until  $\tilde{\mathbf{B}}_{vjq_\rho}$  in row  $q_\rho$ . Therefore the argument that we cannot express row  $\ell \in \{1, \dots, q_\rho\}$  by some linear combination of the rows  $q_\rho + 1, \dots, k + q_\rho$  holds for all  $\ell \in \{1, \dots, q_\rho\}$ . Since  $\mathbf{M}_{ZZ,nT}$  is symmetric this also holds for the columns. Hence the rank of the  $k + q_\rho \times k + q_\rho$  matrix  $\mathbf{M}_{ZZ,n}$  is  $k + q_\rho$  (almost surely).

If  $W_{ij} \neq 0$  for some diagonal element(s), then  $K_{ij} = [(\mathbf{I} - \rho\mathbf{W})]_{ij}$  has some non-zero off diagonal element(s) as well.  $\beta \neq \mathbf{0}_{k \times 1}$  makes the terms  $\beta' \tilde{\mathbf{B}}_{vj}$  non-zero (almost surely). Thus, in the same way we have shown that the rank of  $\mathbf{M}_{ZZ,n}$  is  $k + q_\rho$ , we can show that the rank of the  $k + 1 \times k + q_\rho$  matrix  $\mathbf{M}_{ZZ,n}$  is  $k + 1$  (almost surely).  $\square$

## C Data

*CDS Data:* We use the CDS dataset already used in Schneider et al. (2010), which was obtained from the *Markit Group*. After concentrating on the US market only and by excluding firms with a too large percentage of missing values, 278 firms had been used. The data set also includes the beginning of the financial crises.

*Firm specific and industry data:* To estimate a model, where the default probabilities are driven by firm and industry factors, the following data has been downloaded from *Thomson Datastream* and *Compustat*: (i) Share prices  $p_{it}$  (in US\$) and the number of shares  $NumS_t$ . The Value of preferred stock  $PS_{it}$ , where quarterly records are available. To get weekly data we follow literature and perform linear interpolations. 34 of 176 companies issued preferred stock. In this article we assign preferred stock to equity. Since  $PS_t$  is small compared to debt and the remaining equity, the impact of the assignment to equity is of minor importance, with both the debt to value ratio and the distance to default, respectively. (ii) Short term ( $SD_t$ ) and long term debt ( $LD_t$ ), quarterly records. To get weekly data we follow literature and perform linear interpolations. As mentioned in Section 5.1, matching data from these different data sources provides us with 176 firms.

In addition the following data was collected: (iii) US treasury yields  $r_{.t}$  for the maturities 1, 2, 3, 5, 7, 10 and 30 years (in percentage terms). (iv) Data of the VIX index which is a volatility index obtained from implied Black-Scholes volatilities from the US stock market (for a description see <http://www.cboe.com/micro/VIX/vixintro.aspx>). (v) NAICs industry classification codes. (vi) Standard and Poors (S&P) ratings. (vii) Input-Output data from the BLS Employment Projection Program ([http://www.bls.gov/emp/ep\\_data\\_input\\_output\\_matrix.htm](http://www.bls.gov/emp/ep_data_input_output_matrix.htm)). We excluded firms where we detected further data problems (e.g. extreme spikes, missing values, unclear industry), such that  $N = 148$  firms were still remaining.

From the above balance sheet and stock market data we calculate the *debt to value ratio* measured in percentage terms:

$$DVR_{it} = \left[ \frac{D_{it}}{S_{it} + D_{it}} \right] \cdot 100 , \quad (74)$$

where  $S_{it} = p_{it}NumS_{it} + PS_{it}$  is the *market capitalization* and  $D_{it} = SD_{it} + LD_{it}$  is the *market value of a firm's debt*. As usual in industry and applied academic research we assume that the market value of a firm's debt is equal to the corresponding book value available in the firm's balance sheets.

In Merton type models and in the financial industry the distance to default is frequently used to forecast the conditional probability of default. Intuitively, the distance to default is the number of standard deviations of the annual asset growth by which the firm's expected assets at a given maturity exceed a measure of book liabilities. The distance to default is usually derived by a calibration procedure that matches both market value of equity and equity volatility to the figures that can be observed in the market (for details see Crosbie and Bohn (2003)). In this paper the distance to default is derived from

$$DD_{it} = \frac{VA_{it} - DP_{it}}{VA_{it}\sigma_{Ait}} . \quad (75)$$

$VA_{it}$  is the firm value. The *default point*  $DP_{it}$  is the sum of short-term liabilities +1/2 long-term liabilities, i.e.  $DP_{it} = SD_{it} + 1/2LD_{it}$ .  $\sigma_{Ait}$  is the standard deviation of the firm value;  $\sigma_E$  is a measure of the equity volatility. Based on Crosbie and Bohn (2003):

$$\begin{aligned} VA_{it} &= VE_{it}\mathcal{N}(d_{1i}) + \exp(-y_{tm}M)(SD_{it} + LD_{it})\mathcal{N}(d_{2i}) , \\ \sigma_{Ait} &= \sigma_{Ei} \frac{VE_{it}}{VA_{it}} , \\ d_{1i} &= \frac{\log(VA_{it}/(SD_{it} + LD_{it})) + (y_{tm} + \frac{1}{2}\sigma_{Ait}^2)M}{\sqrt{\sigma_{Ait}^2M}} , \\ d_{2i} &= d_{1i} - \sqrt{\sigma_{Ait}^2M} . \end{aligned} \quad (76)$$

The standard deviation of the firm value,  $\sigma_{Ait}$ , is derived by an implicit estimation from the Black/Scholes formula. We derived estimates of  $VA_{it}$  and  $\sigma_{Ait}$  by minimizing a weighted sum of the squared distances between the model implied value of equity  $VE_{it}$  and the market capitalization  $S_{it}$ , and the terms  $\sigma_{Ait}VA_{it}$  and  $\sigma_{Ei}VE_{it}$ , respectively. Following industry praxis we set  $M = 1$  and  $y_{tm}$  equal to the one year treasury yield  $r_{1t}$ . We have to point out that the minimization strongly depends on how all these values are scaled.

$\sigma_{E_i}^2$  is estimated from log asset returns. Here e.g. the sample variance  $\hat{\sigma}_{iE}^2$  (resulting in a constant equity volatility) can be used. In this article we follow Ericsson et al. (2009) and approximate the equity volatility by means of exponential smoothing:

$$\sigma_{E,it}^2 = \lambda \sigma_{E,it-1}^2 + (1 - \lambda) (\Delta \log p_{it})^2 \quad (77)$$

with  $\lambda = 0.94$ . This  $\sigma_{E,it}^2$  has been used in the calculation of the distance to default.

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## D Questions raised and ...

1. Why does *2SLS* work? Effects the long run covariance the volatility of the estimator make the size not bad.
2. Why is *DOLS* not sufficient: "spatial endogeneity"
3. Why is *2SLS* not sufficient:  $\tilde{\mathbf{x}}_{it}$  is an element of  $\mathbf{Z}_{it}$ , this part of the instruments is still correlated with the noise  $u_{it}^\dagger$
4. Can we use  $\tilde{\mathbf{z}}_{it} = (\tilde{\mathbf{x}}_{it}^*, \tilde{\mathbf{x}}'_{jt})'$ , where  $\tilde{\mathbf{x}}_{it}^* = \sum_{j=1}^n W_{ij} \tilde{\mathbf{x}}_{jt}$  as instruments. Then we need not use *DOLS*. With this set of instruments it is not clear to me under what conditions the matrix  $Z'Z$  remains regular. A counterexample with a singular matrix is easy to construct. Suppose that  $k = q_\rho = 1$  and  $n = 2$ . Then  $x_{1t} = (W_{12}y_{2t}, x_{1t})'$ ,  $z_{1t} = (W_{12}x_{2t}, x_{2t})'$ ,  $x_{2t} = (W_{21}y_{1t}, x_{2t})'$ ,  $z_{2t} = (W_{21}x_{1t}, x_{2t})'$ . Then

$$\begin{aligned} \mathbf{M}_{ZZ,n} &= \frac{1}{n} \sum_{i=1}^n \lim_{t \rightarrow \infty} \frac{1}{T_\star^2} \sum_{t=1}^{T_\star} \mathbf{Z}_{it,1:d_z} \mathbf{Z}'_{it,1:d_z} \\ &= \left( \begin{array}{c|c} \int W_{12}^2 \mathcal{B}_2^2 & W_{12} \mathcal{B}_2^2 \\ \hline \int W_{12} \mathcal{B}_2^2 & \mathcal{B}_2^2 \end{array} \right) + \left( \begin{array}{c|c} \int W_{21}^2 \mathcal{B}_1^2 & W_{21} \mathcal{B}_1^2 \\ \hline \int W_{21} \mathcal{B}_1^2 & \mathcal{B}_1^2 \end{array} \right). \end{aligned} \quad (78)$$

Here  $\mathbf{M}_{ZZ,n}$  is not of rank  $k + q_\rho$ .

5. Consider (1) for the two-dimensional case, i.e.  $n = 2$ . Due to Assumption 1 the matrix  $\mathbf{I}_2 - \rho \mathbf{W}$  has to be invertible, such that

$$\begin{pmatrix} y_{1t} \\ y_{21t} \end{pmatrix} = \frac{1}{1 + \rho^2 W_{12} W_{21}} \cdot \begin{pmatrix} \beta^\top \mathbf{x}_{1t} & -\rho W_{21} \beta^\top \mathbf{x}_{2t} & +u_{1t}^\dagger & -\rho W_{21} u_{2t}^\dagger & +\alpha_1 \\ -\rho W_{12} \beta^\top \mathbf{x}_{1t} & +\beta^\top \mathbf{x}_{2t} & +u_{2t}^\dagger & -\rho W_{12} u_{1t}^\dagger & +\alpha_1 \end{pmatrix}. \quad (79)$$

In (79) we observe that  $u_{1t}^\dagger$  is correlated with  $y_{2t}$  and vice versa. Therefore  $W_{12}y_{2t}$  is correlated with the noise term. The instrument in this case is  $W_{12}\tilde{\mathbf{x}}_{2t}$ .  $W_{12}\tilde{\mathbf{x}}_{2t}$  is correlated with  $\tilde{y}_{2t}$  (if  $\beta \neq 0$ ). The

terms in  $\tilde{y}_{2t}$  correlated with  $\tilde{\mathbf{x}}_{2t}$  are  $\tilde{\mathbf{x}}_{2t}$  and  $u_{2t}^\dagger \cdot \frac{1}{T^*} \sum \tilde{\mathbf{x}}_{2t} \tilde{u}_{2t}$  converges to  $\Omega_{u,22} \cdot \int \tilde{\mathcal{B}}_2 d\mathcal{W}_2 + \Lambda_{u2,x2} = \Omega_{u,22} \cdot \int \tilde{\mathcal{B}}_2 d\mathcal{W}_2 + \Gamma_{uv}$  (Davidson (1994)[Theorem 30.13]). In the terms considered these terms are scaled by  $(\frac{1}{T^*})^2$ . Therefore this term goes to zero.