A Geometric Treatment Of Time Varying Volatilities

Chulwoo Han

May 2015

Table Of Contents

Motivation

- The covariance matrix is the central part of many financial theories and models:
  - ✓ Portfolio optimization;
  - ✓ Credit risk models;
  - ✓ Asset pricing.
- Adequate models for covariance dynamics are still lacking:
  - ✓ The covariance matrix is normally assumed constant or to follow a linear process;
  - ✓ Observed covariances are, on the other hand, often time varying and show nonlinear behavior, especially under extreme economic conditions.

Motivation

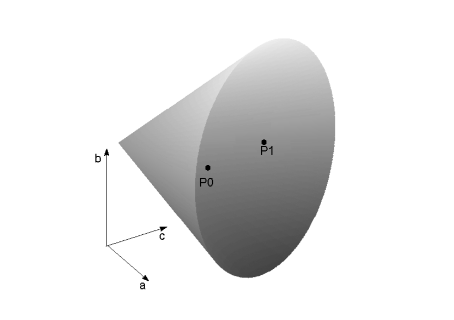
- Two fundamental problems arising in modeling and estimation of covariance dynamics models are:
  - ✓ Preserving positive definiteness;
  - ✓ Curse of dimensionality.
- Earlier treatments of these problems are rather *ad hoc*:
  - ✓ None of them seem to achieve parsimoniousness and positive definiteness in a formal way;
  - ✓ The parameters often lack intuitive meaning;
- These problems can be traced to the fact that earlier models fail to respect the geometric properties of the covariance matrix.

Importance of Geometry: An Illustrative Example

Let $P_0, P_1 \in P(2)$ be two symmetric, positive definite 2×2 real matrices:

$$P_0 = \begin{pmatrix} a_0 & b_0 \\ b_0 & c_0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \tag{1}$$

with $a_i c_i - b_i^2 > 0$ and $a_i > 0$.
Suppose we want to construct a straight line connecting P_0 and P_1.

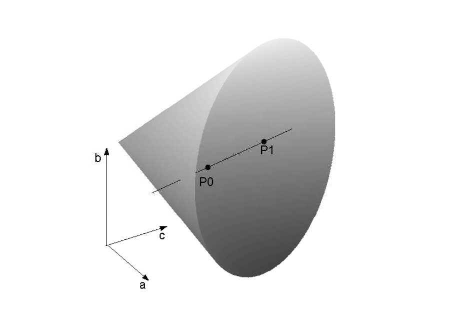


Importance of Geometry: An Illustrative Example

- A Naive Way:

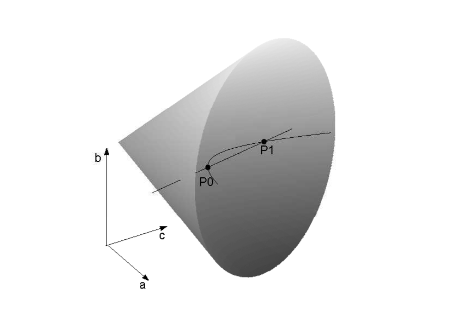
$$P(t) = (1-t)P_0 + tP_1 \tag{2}$$

✓ A problem with this approach is the straight line may not remain within the space $P(2)$.



Importance of Geometry: An Illustrative Example

- A Better Way:
  - ✓ If we define a proper metric on $P(2)$, the minimal geodesic (the shortest distance path) can be obtained.
  - ✓ The minimal geodesic always lies within $P(2)$.



Geometry of $P(n)$

- The covariance space $P(n)$ is defined as

$$P(n) = \left\{ P \in \mathbb{R}^{n \times n} \mid P = P^\top, P > 0 \right\}. \tag{3}$$

- $P(n)$ is a differentiable manifold whose tangent space at a point $P \in P(n)$ can be identified with $n \times n$ symmetric matrices $S(n)$.
- A Riemannian structure can be constructed via the Riemannian metric given by $\langle X, Y \rangle_P = \mathrm{tr}(P^{-1}XP^{-1}Y)$.
- In terms of this metric, the length of a curve $P(t) \in P(n)$, $a \leq t \leq b$, is given by

$$L(P) = \int_a^b \sqrt{\mathrm{tr}\left((P^{-1}(t)\dot{P}(t))^2 \right)}\, dt. \tag{4}$$

Geometry of $P(n)$

- **The Minimal Geodesic** $\gamma(t) : [0,1] \to [A,B],\ A,B \in P(n)$

$$\gamma(t) = G\left(G^{-1}BG^{-\top}\right)^{t}G^{\top} \tag{5}$$

where $GG^{\top} = A,\ \ G \in GL^{+}(n)$.

- **Riemannian Log Map**
The tangent vector of the geodesic at A:

$$\mathrm{Log}_A(B) = G\log\left(G^{-1}BG^{-\top}\right)G^{\top}. \tag{6}$$

- **Riemannian Exponential Map**
The minimal geodesic emanating from $A \in P(n)$ in the direction X:

$$\mathrm{Exp}_A(X) = G\exp\left(G^{-1}XG^{-\top}\right)G^{\top}. \tag{7}$$

Geometry of $P(n)$

- **Distance**
  Defining the distance between A and B in the usual way by the length of the above minimal geodesic, we have

$$d(A,B) = \left(\sum_{i=1}^{n} \log^2 \lambda_i \right)^{1/2}, \tag{8}$$

  where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix AB^{-1}.

- **Intrinsic mean**

$$\arg\min_{\bar{P} \in P(n)} \sum_{i=1}^{N} d(\bar{P}, P_i)^2. \tag{9}$$

Principal Geodesic Analysis

- While principal component analysis (PCA) seeks the principal axes of variation in Euclidean space, **principal geodesic analysis** (PGA) seeks a submanifold that best represents the variability of the data in a Riemannian manifold.
- It can be shown that the PGA can be performed by applying the PCA to the tangent space of the manifold, $T_\mu M$.
- Given principal directions, V_k, a point in $P(n)$ can be generated by the formula

$$P = \mathrm{Exp}_{\bar{P}}\left(\sum_{k=1}^{K} \alpha_k V_k\right),$$

for some α_k and $K \leq n(n+1)/2$.

Covariance Dynamics

- **System Equation**

$$y_t = \mu + e_t, \qquad e_t \sim N(0, H_t), \tag{10}$$

- **Dynamics of H_t**

$$dH_t = F_t dt \tag{11}$$

where F_t is a time-varying $n \times n$ symmetric matrix which depends on the information set at t.

- The minimal geodesics provide a natural way of discretizing general differential equations on $P(n)$.

$$H_t = \mathrm{Exp}_{H_{t-1}}(F_t). \tag{12}$$

- Another class of dynamics we consider.

$$H_t = \mathrm{Exp}_{H_\infty}(F_t). \tag{13}$$

A Geometric GARCH Model

F_t can be defined as a function of lagged terms of covariance and residuals.

$$\begin{aligned} F_t &= \sum_{p=1}^{P}(A_p H_{t-p} + H_{t-p} A_p^\top) \\ &\quad + \sum_{q=1}^{Q}(B_q e_{t-q} e_{t-q}^\top + e_{t-qq} e_{t-q}^\top B_q^\top) \\ &\quad + \sum_{r=1}^{R}(D_r \eta_{t-r} \eta_{t-r}^\top + \eta_{t-r} \eta_{t-r}^\top D_r^\top). \end{aligned}$$

where $\eta_t = |e_t| - e_t$.
We call this specification of time varying volatilities the **geometric GARCH** or simply **GGARCH** model.

PCA Based Specifications

The PCA can be applied in two ways:

- Usual PCA to the tangent vectors connecting H_{t-1} and H_t.
- PGA to the tangent vectors connecting H_∞ and H_t.

F_t can be written in the form

$$F_t = \sum_{k=1}^{K} \alpha_{kt} V_k$$

$$\begin{aligned}\alpha_{kt}(H_{t-1}, e_{t-1}, \eta_{t-1}) &= a_k^\top vech(H_{t-1}) + b_k^\top vech(e_{t-1}e_{t-1}^\top) \\ &\quad + c_k^\top vech(\eta_{t-1}\eta_{t-1}^\top)\end{aligned}$$

Or

$$\alpha_{kt}(H_{t-1}, e_{t-1}, \eta_{t-1}) = d(f_k(H_{t-1}, e_{t-1}, \eta_{t-1}), H_{t-1})$$

$$f_k(H_{t-1}, e_{t-1}, \eta_{t-1}) = a_k H_{t-1} + b_k e_{t-1}e_{t-1}^\top + c_k \eta_{t-1}\eta_{t-1}^\top$$

Parsimonious Representations

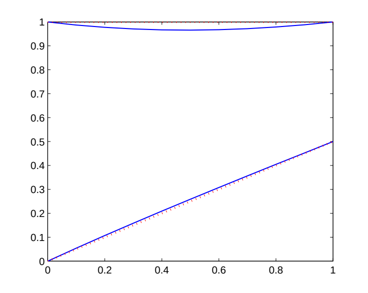
While BEKK and DCC models retain n^2 term regardless of the simplicity of the model, diagonal or simpler representations of the GGARCH models have parameter numbers of $O(n)$ or a constant.

Model	Parameters	Description
GGARCH PCA DIST	$3K$	α_{kt} is defined by the distance function.
GGARCH PCA DIAG	$3nK$	Off-diagonal elements are ignored.
GGARCH PCA FULL	$(1.5n^2 + n)K$	All elements are considered.
GGARCH SCALAR	3	Coefficient matrices are scalar.
GGARCH DIAG	$3n$	Coefficient matrices are diagonal.
GGARCH LINEAR	$1.5n^2 + 1.5n$	Coefficient matrices are symmetric and the matrix products are element-wise.
GGARCH FULL	$3n^2$	Coefficient matrices are arbitrary $n \times n$.
BEKK SCALAR	$0.5n^2 + 0.5n + 3$	BEKK model with scalar coefficients.
BEKK DIAGONAL	$0.5n^2 + 3.5n$	BEKK model with diagonal coefficients.
BEKK FULL	$3.5n^2 + 0.5n$	BEKK model with arbitrary $n \times n$ coefficients.
DCC 3-STAGE	$n^2 + 4n + 3$	DCC model.

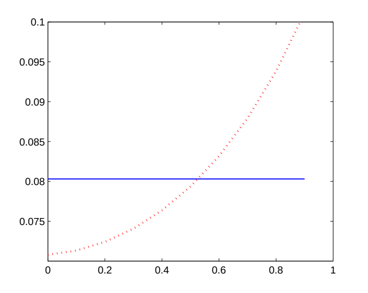
The Shortest Path

Consider the trajectory between two covariance matrices

$$H_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } H_1 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$



(a) Trajectory of $H(t)$



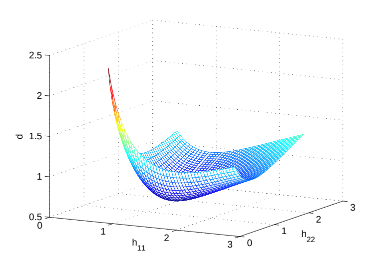
(b) Distance $(H(t-0.1), H(t))$

In (a), trajectory of variance at the top and trajectory of covariance at the bottom. Solid lines: geodesic, dotted lines: linear interpolation.

The Shortest Path

- The trajectory of the variance is convex and that of the covariance is slightly concave, contrary to the naive linear interpolation that yields straight lines.
- Under the Riemmanian metric, distance between two covariance matrices increases exponentially as one matrix approaches singularity, *i.e.*, perfect correlation. This is a desirable property as one would consider correlation increase from 0.0 to 0.5 more probable than increase from 0.5 to 1.0.
- The distance between two intermediate points on the geodesic is constant, while that on the linearly interpolated line increases as t increases, eventually resulting in a longer distance between H_0 and H_1.

The Shortest Path



Distance between H_0 and H_1. h_{11} and h_{22} are variances of H_1.

- The minimum distance is achieved when the variances are about 1.275: A conventional metric would have the minimum distance when the variances are 1.
- An economic explanation: When the correlation between two variables increases, the variances are also likely to increase due to positive feedback.

Riemmanian Exponential Map

- Initial covariance matrix

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

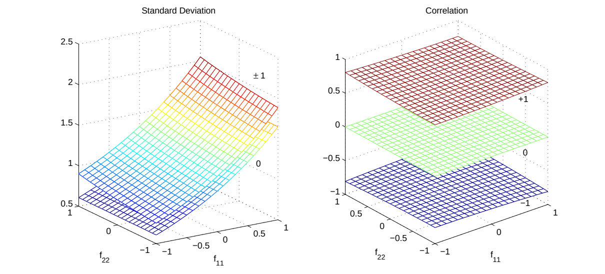
- Tangent vector

$$H = \mathrm{Exp}_H(F), \quad F = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{11} \end{bmatrix}$$

with $f_{11} = \{-1, -0.9, \ldots, 1\}$, $f_{22} = \{-1, -0.9, \ldots, 1\}$, $f_{12} = \{-1, 0, 1\}$.

Riemmanian Exponential Map

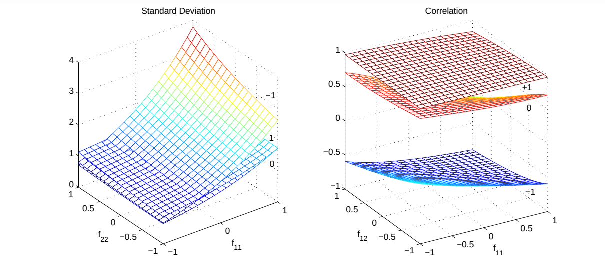
$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



- Variances increase exponentially with f_{ii}: can be related to rapid market destabilization.
- No influence of f_{22} on h_{11} when uncorrelated.

Riemmanian Exponential Map

$$H = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$



- H is more sensitive to F, especially when $f_{12} = -1$.
- Shocks of opposite direction rapidly reduce correlation.

Global Market Correlation

The GGARCH models are applied to S&P500 and FTSE100 daily returns and compared with BEKK and DCC models. Sample period is from October 1, 2003 to September 30, 2013.

	Mean	Covariance		Correlation	
		S&P500	FTSE100	S&P500	FTSE100
S&P500	1.924E-04	1.600E-04	0.872E-04	1.000	0.578
FTSE100	1.680E-04	0.872E-04	1.425E-04	0.578	1.000

Global Market Correlation: Test Models

$$r_t = \mu + e_t, \qquad e_t \sim N(0, H_t),$$
$$H_t = \mathrm{Exp}_{H_{t-1}}(F_t).$$

- GGARCH SCALAR

$$F_t = AH_{t-1} + Be_{t-1}e_{t-1}^\top + C\eta_{t-1}\eta_{t-1}^\top$$

where A, B, and C are scalar.

- GGARCH DIAG

$$\begin{aligned} F_t &= AH_{t-1} + H_{t-1}A^\top + Be_{t-1}e_{t-1}^\top + e_{t-1}e_{t-1}^\top B^\top \\ &\quad + C\eta_{t-1}\eta_{t-1}^\top + \eta_{t-1}\eta_{t-1}^\top C^\top \end{aligned}$$

where A, B, and C are diagonal.

Global Market Correlation: Test Models

- GGARCH LINEAR

$$F_t = A \otimes H_{t-1} + B \otimes e_{t-1}e_{t-1}^\top + C \otimes \eta_{t-1}\eta_{t-1}^\top$$

where A, B, and C are symmetric, and $\otimes$ is the element-wise matrix product.

- GGARCH FULL

$$\begin{aligned} F_t &= AH_{t-1} + H_{t-1}A^\top + Be_{t-1}e_{t-1}^\top + e_{t-1}e_{t-1}^\top B^\top \\ &\quad + C\eta_{t-1}\eta_{t-1}^\top + \eta_{t-1}\eta_{t-1}^\top C^\top \end{aligned}$$

where A, B, and C are arbitrary $n \times n$ matrices.

Global Market Correlation: Test Models

- GGARCH PCA DIAG

$$F_t = \sum_{k=1}^{K} \alpha_{kt} V_k$$

$$\alpha_{kt} = a_k^\top diag(H_{t-1}) + b_k^\top diag(e_{t-1}e_{t-1}^\top) + c_k^\top diag(\eta_{t-1}\eta_{t-1}^\top)$$

where a_k, b_k, and c_k are $n \times 1$ vectors.

- GGARCH PCA FULL

$$F_t = \sum_{k=1}^{K} \alpha_{kt} V_k$$

$$\alpha_{kt} = a_k^\top vech(H_{t-1}) + b_k^\top vech(e_{t-1}e_{t-1}^\top) + c_k^\top vech(\eta_{t-1}\eta_{t-1}^\top)$$

where a_k, b_k, and c_k are $n(n+1)/2 \times 1$ vectors.

Global Market Correlation: PCA

Covariance matrix time series for PCA-based GGARCH models are generated from sample covariance of the minimum size (two) subsample at each time t.

- The first component is related to simultaneous change of the variance and covariance;
- The second component is related to independent change of variances;
- The third component is related to independent change of the covariance.

	1st component	2nd component	3rd component
Eigenvalue	1.703E-06	0.375E-06	0.092E-6
	(78.479%)	(17.281%)	(4.240%)
	0.6470	0.6938	0.3163
Eigenvector	0.4972	-0.0693	-0.8649
	0.5781	-0.7168	0.3898

Global Market Correlation: Estimation and Diagnosis

	Log-Likelihood
GGARCH SCALAR	1.7353E+04
GGARCH DIAG	1.7356E+04
GGARCH LINEAR	1.7409E+04
GGARCH FULL	1.7371E+04
GGARCH PCA DIAG	1.7404E+04
GGARCH PCA FULL	1.7478E+04
BEKK SCALAR*	1.7479E+04
BEKK DIAGONAL**	1.7509E+04
DCC 3-STAGE***	1.7554E+04

Table : Log-likelihood as a result of QMLE.

Global Market Correlation: Estimation and Diagnosis

	Q	p-value
GGARCH SCALAR	76.8857	0.0000
GGARCH DIAG	71.4855	0.0004
GGARCH LINEAR	37.5462	0.0646
GGARCH FULL	65.4737	0.0012
GGARCH PCA DIAG	70.6275	0.0000
GGARCH PCA FULL	41.8565	0.0060
BEKK SCALAR*	21.7602	0.1427
BEKK DIAGONAL**	17.2584	0.2642
DCC 3-STAGE***	14.5448	0.4504

Table : Ljung-Box autocorrelation test results. Q denotes Ljung-Box Q statistic.

Global Market Correlation: Estimation and Diagnosis

	σ_r	w_1	w_2
GGARCH SCALAR***	0.0109	0.5079	0.4921
GGARCH DIAG	0.0110	0.5051	0.4949
GGARCH LINEAR	0.0110	0.5011	0.4989
GGARCH FULL	0.0110	0.5134	0.4866
GGARCH PCA DIAG	0.0110	0.5118	0.4882
GGARCH PCA FULL	0.0110	0.4861	0.5139
BEKK SCALAR	0.0110	0.5232	0.4768
BEKK DIAGONAL	0.0110	0.5335	0.4665
DCC 3-STAGE	0.0110	0.5268	0.4732

Table : Minimum variance portfolio. σ_r is the sample standard deviation of the portfolio return, and w_1 and w_2 are average portfolio weights of S&P500 and FTSE100, respectively.

Global Market Correlation: Estimation and Diagnosis

	20:80			50:50			80:20		
	0.05	0.01	0.001	0.05	0.01	0.001	0.05	0.01	0.00
GGARCH SCALAR*	0.0518	0.0180	0.0042	0.0541	0.0161	0.0050	0.0525	0.0188	0.004
GGARCH DIAG	0.0525	0.0165	0.0042	0.0541	0.0165	0.0050	0.0521	0.0196	0.005
GGARCH LINEAR	0.0548	0.0184	0.0046	0.0567	0.0180	0.0058	0.0579	0.0215	0.005
GGARCH FULL	0.0529	0.0180	0.0042	0.0533	0.0176	0.0050	0.0560	0.0188	0.004
GGARCH PCA DIAG***	0.0514	0.0153	0.0035	0.0479	0.0138	0.0038	0.0502	0.0150	0.002
GGARCH PCA FULL**	0.0548	0.0192	0.0046	0.0521	0.0207	0.0035	0.0506	0.0188	0.003
BEKK SCALAR	0.0537	0.0176	0.0031	0.0556	0.0192	0.0038	0.0541	0.0199	0.005
BEKK DIAGONAL	0.0571	0.0207	0.0042	0.0579	0.0215	0.0046	0.0560	0.0219	0.005
DCC 3-STAGE	0.0602	0.0211	0.0042	0.0590	0.0226	0.0050	0.0575	0.0226	0.005

Table : Value-at-Risk test. The first row indicates portfolio composition between S&P500 and FTSE100, and the second row indicates probability level. The figures are probability of loss exceeding VaR.

Global Market Correlation: Summary

- The results are mixed and do not consistently support any particular model.
- This could be a limitation of the specific model under consideration, or evidence of fundamental limitation of our geometric framework.
- Positive side is that, while BEKK and DCC models perform better in term of in-sample fitting, the GGARCH models are better performers for future risk estimation.
- Since the covariance matrix evolves in an exponential manner, the covariance matrix more often than not becomes numerically unstable, in which case we assign an arbitrary large number of log-likelihood value. This interruption may cause a sub-optimal estimation results.

Summary

- We have proposed a new framework for addressing the covariance dynamics.
- It preserves geometric structure of the covariance matrix without any arbitrary restricions by respecting the inherent geometric features of the covariance matrix.
- It also seems to possess the desired nonlinear natures of the covariance dynamics observed in the market.
- Empirical studies reveal the potential for the growth of our model by showing that our model does capture many well-known features about volatility transmission between markets.

Directions for Future Research

- More comprehensive empirical studies and comparison analysis with other models are in order.
- Numerous areas of application can be sought: credit risk modeling, asset pricing, portfolio optimization, etc.
- Econometric methods to address the significance of the estimated parameters are yet to be established.
- The framework can be extended to multivariate normal distributions.
- In a broad context, the framework presents a new approach to treating nonlinear properties observed in the financial market and can contribute to building a new paradigm for economic modeling.

Thank you for your attention.