Conditional Expected Utility Criteria for Decision Making under Ignorance or Objective Ambiguity

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PRELIMINARY AND INCOMPLETE DRAFT, NOT TO BE QUOTED NOR CIRCULATED

Abstract

We provide an axiomatic characterization of a family of criteria for ranking *completely uncertain* and/or *ambiguous* decisions. A completely uncertain decision is described by the set of all its consequences (assumed to be finite). An ambiguous decision is described as a finite set of possible probability distributions over a finite set of prices. Every criterion in the family compares sets on the basis of their *conditional expected utility*, for some probability function taking strictly positive values and some utility function both having the universe of alternatives as their domain.

1 Introduction

To be provided

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2 The Model

2.1 Notation

The sets of integers, non-negative integers, real numbers and non-negative real numbers are denoted respectively by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{R}_+$. If v is a vector in $\mathbb{R}^k$ for some strictly positive integer k and α is a real number, we denote by $\alpha.v$ the scalar product of α and v. Our notation for vectors inequalities is $\geqq$, $\geq$ and $>$. By a *binary relation* $\succsim$ on a set Ω, we mean a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in R$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \Longleftrightarrow x \succsim y$ and $y \succsim x$ and its *asymmetric* factor $\succ$ by $x \succ y \Longleftrightarrow x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every x in Ω, is *transitive* if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any $x, y, z \in \Omega$ and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in Ω. An equivalence class C of a binary relation $\succsim$ on Ω is a subset of Ω such that $c \sim c'$ for all c, $c' \in C$ and it is not the case that $c \sim c'$ if $c \in C$ and $c' \in \Omega \backslash C$. A reflexive, transitive and complete binary relation is called an *ordering*. An ordering is *trivial* if it has only one equivalence class.

2.2 Basic concepts

Let X be the set of consequences. While we do not make any specific assumptions on X, it will be clear subsequently that the axioms that we impose makes it natural to regard this set as infinite. As an example, one could think of X as being $\mathbb{R}$, interpreted as the set of all conceivable financial returns (either negative or positive) of some investment decision in a highly uncertain environment, even though we will see that this setting has rather specific implication on the ranking of uncertain decisions that it allows. As another example, one could think of X as the set of all conceivable probability distributions on a basic set of k different prices.

We denote by $\mathcal{P}(X)$ the set of all non-empty *finite* subsets of X (with generic elements A, B, C, etc.). Any such a subset is interpreted as a description of all consequences of an *uncertain* decision or, for short, as a *decision*. A *certain* decision with consequence $x \in X$ is identified by the singleton $\{x\}$.

Let $\succsim$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be an *ordering* on $\mathcal{P}(X)$. We interpret the statement $A \succsim B$ as meaning "decision with consequences in A is weakly preferred to decision with consequences in B". A similar interpretation is given to the statements $A \succ B$ ("strictly preferred to") and $A \sim B$ ("indifference").

We want to identify the properties (axioms) of the ordering $\succsim$ that are necessary and sufficient for the existence of a function $u : X \to \mathbb{R}$ and a function $p : X \to \mathbb{R}_{++}$ that are such that that, for every A and B in $\mathcal{P}(X)$:

$$A \succsim B \iff \frac{\sum_{a \in A} p(a)u(a)}{\sum_{a \in A} p(a)} \geq \frac{\sum_{b \in B} p(b)u(b)}{\sum_{b \in B} p(b)}. \tag{1}$$

We refer to any ordering numerically represented as per (1) for some functions p and u as to a Conditional Expected Utility (CEU) criterion. Indeed, the function p is naturally interpreted as a "probability" function that assigns to each consequence (or lottery in the objective ambiguity framework) a number that reflects its *a priori* "likelihood", while the u function is interpreted as a conventional utility function that evaluates the "desirability" of every consequence from the subjective viewpoint of the decision maker. Hence an ordering represented by (1) can therefore be seen as comparing decisions under ambiguity or ignorance on the basis of the expected utility of the consequences of these decisions conditional upon the fact that they will materialize.

We notice that the family of Uniform Expected Utility criteria characterized in [2] is, a priori, a subclass of this family, in which the function p is any constant function. Yet, as we shall see later, the characterization that we provide of this family is not complete as it does not cover all criteria that belong to the family represented by (1). The reason for this is that we characterize this family by assuming that both the universe X and the ordering $\succsim$ satisfies the following "richness" condition (considerably stronger than the condition of the same name used in [2]).

A 1 Richness. *For every $E' \sim A \prec B \prec C \sim E$, there exists D and D' satisfying $D \cap (A \cup C \cup E) = \varnothing = D' \cap (A \cup C \cup E')$ such that $D \sim C$, $D' \sim A$ and $D \cup A \sim B \sim D' \cup C$.*

The axioms that characterize this family of criteria (in an environment that satisfies this richness) are the following.

A 2 Balancedness. *Suppose A, B, C and D are such that $(A \cup B) \cap (C \cup D) = \varnothing$ and $A \cup C \succsim B \cup C$. If $A \sim B \succ C, D$, then $A \cup D \succsim B \cup D$.*

A 3 Averaging. *Suppose A and B are disjoint. Then $A \succsim B$ iff $A \cup B \succsim B$ iff $A \succsim A \cup B$.*

A 4 Archimedean. *Suppose $A \sim B \sim C \sim D \not\sim F$, $A \cup F \succ B \cup F$ and $F \cap (C \cup D) = \emptyset$. If there exists two infinite sequences $A_0, A_1, \ldots, A_i, \ldots$ and $B_0, B_1, \ldots, B_i, \ldots$, with $A_i \cap (F \cup C \cup A_j) = \emptyset$, $B_i \cap (F \cup D \cup B_j) = \emptyset$, $A_i \sim A$, $B_i \sim B$, $A_i \cup F \sim A \cup F$ and $B_i \cup F \sim B \cup F$ for all $i \neq j \in \mathbb{N}$, then $C \cup F \bigcup_{i=0}^{n} A_i \succsim D \cup F \bigcup_{i=0}^{n} B_i$ for some $n \in \mathbb{N}$.*

SOME COMMENTS MUST BE ADDED HERE TO EXPLAIN THE AXIOMS.

3 Main results

Lemma 1 *Let $\succsim$ be a non-trivial ordering of $\mathcal{P}(X)$ satisfying Richness and Averaging. Then, for every $A, C \in \mathcal{P}(X)$, there exists $B \in \mathcal{P}(X)$ such that $B \sim A$ and $B \cap (A \cup C) = \varnothing$.*

Proof. Because of non-triviality, we know that there is a set D such that $D \prec A$ or $A \prec D$. We treat the case $A \prec D$ (the other case is handled symmetrically). We first prove that there are at least two equivalence classes better than the one containing A, so that it will be possible to apply Richness. We consider two cases: (1) $A \cap D = \varnothing$. Then Averaging yields $A \prec A \cap D \prec D$ (and we are done). (2) $A \cap D \neq \varnothing$. We consider three subcases: (a) $A \cap D \sim A$. Then, by Averaging, $A \setminus D \sim A$ and $A \setminus D \prec A \cup D \prec D$. (b) $A \cap D \prec A$. Then Averaging implies $D \prec D \setminus A$. (c) $A \cap D \succ A$. If $A \cap D \not\sim D$, then we are done. Otherwise, by Averaging, $D \setminus A \sim D$ and $A \prec A \cup (D \setminus A) \prec D \setminus A$.

A first application of Richness yields a set B_1 such that $B_1 \sim A$ and $B_1 \cap A = \varnothing$. If $B_1 \cap C = \varnothing$, then the proof is done. If $B_1 \cap C \neq \varnothing$, then use Richness again to find a set B_2 such that $B_2 \sim A \cup B_1$ and $B_2 \cap (A \cup B_1) = \varnothing$. By Averaging, $A \cup B_1 \sim A$ and, by transitivity, $B_2 \sim A$. We are now sure that B_2 does not contain any of the elements of $B_1 \cap C$. If $B_2 \cap C = \varnothing$, then the proof is done. If $B_2 \cap C \neq \varnothing$, then use Richness again to find a set B_3 such that $B_3 \sim A \cup B_1 \cup B_2$ and $B_3 \cap (A \cup B_1 \cup B_2) = \varnothing$. By Averaging, $A \cup B_1 \cup B_2 \sim A$ and, by transitivity, $B_3 \sim A$. Notice that $(B_1 \cup B_2) \cap C \supsetneq B_1 \cap C$ We are now sure that B_3 does not contain any of the elements of $(B_1 \cup B_2) \cap C$. If $B_3 \cap C = \varnothing$, then the proof is done. If $B_3 \cap C \neq \varnothing$, we iterate this construction and we find $B_4, B_5, \ldots$ At each iteration, $(B_1 \cup \ldots \cup B_i) \cap C \supsetneq (B_1 \cup \ldots \cup B_{i-1}) \cap C$. Since C is finite, we are sure to reach some B_j satisfying the same conditions as B in the statement of the lemma. □

Let us define the sets $m(X)$ and $M(X)$ of minimal (resp. maximal) decisions by $m(X) = \{A \in \mathcal{P}(X) : A \precsim B \ \forall B \in \mathcal{P}(X)\}$ and $M(X) = \{A \in \mathcal{P}(X) : A \succsim B \ \forall B \in \mathcal{P}(X)\}$. These sets can be empty. We define $\mathcal{P}_*(X)$ by means of $\mathcal{P}_*(X) = \mathcal{P}(X) \setminus (m(X) \cup M(X))$.

Lemma 2 *If $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness and Averaging, then, for every set $B \in \mathcal{P}_*(X)$, there are $A, C \in \mathcal{P}_*(X)$ such that $A \prec B \prec C$.*

Proof. If $\succsim$ is not trivial, then there are $D, E \in \mathcal{P}(X)$ such that $D \prec E$. By Lemma 1, there is a set $F \in \mathcal{P}(X)$ such that $F \sim D$ and $F \cap (D \cup E) = \varnothing$. By Averaging and Transitivity, $D \prec F \cup E \prec E$. So, $\succsim$ has at least three equivalence classes and, hence, $\mathcal{P}_*(X)$ is not empty. Let B be a decision in $\mathcal{P}_*(X)$ (we have just proved that it exists). We will prove that there is $A \in \mathcal{P}_*(X)$ such that $A \prec B$ (the proof that there is $C \in \mathcal{P}_*(X)$ such that $B \prec C$ is similar). If $m(X)$ is empty, then the proof is immediate. So, we consider that $m(X)$ is not empty. Let G be a decision in $m(X)$. By Lemma 1, there is a set $H \in \mathcal{P}(X)$ such that $H \sim G$ and $H \cap (G \cup B) = \varnothing$. By Averaging and Transitivity, $H \prec H \cup B \prec B$. □

A consequence of this lemma is that $\mathcal{P}(X)$ is infinite and, hence, X is infinite. For any $E \in \mathcal{P}_*(X)$, define $\mathcal{P}^E(X) = \{C \in \mathcal{P}(X) : C \sim E\}$. It is the equivalence class of $\succsim$ containing the set E. Define then the binary relation $\succsim_l$ on $\mathcal{P}^E(X)$ by $A \succsim_l B$ iff there exists C disjoint from A and B such that $A \cup C \succsim B \cup C$ and

$C \prec E$. Notice that we do not define $\succsim_l$on a maximal (or minimal) equivalence class.

ADD SOME COMMENTS ON THE INTUITIVE MEANING OF THE BINARY RELATION $\succsim_l$

Lemma 3 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Balancedness and Averaging. Then the relation $\succsim_l$ is a weak order.*

Proof. Let A, B, C be three sets in $\mathcal{P}^E(X)$ such that $A \succsim_l B$ and $B \succsim_l C$. By definition of $\succsim_l$, this implies the existence of D, D' respectively disjoint from $A \cup B$ and $B \cup C$ such that $E \succ D, D'$, $A \cup D \succsim B \cup D$ and $B \cup D' \succsim C \cup D'$. Thanks to Lemma 1, we choose D'' disjoint from $A \cup B \cup C$, with $D \sim D''$. By Balancedness, $A \cup D'' \succsim B \cup D''$ and $B \cup D'' \succsim C \cup D''$. By transitivity, $A \cup D'' \succsim C \cup D''$ and, hence, $A \succsim_l C$. This proves the transitivity of $\succsim_l$. We now turn to the completeness of $\succsim_l$.

Let A, B be two sets in $\mathcal{P}^E(X)$ such that $A \not\succsim_l B$. By definition of $\succsim_l$, either (i) there is no set C disjoint from $A \cup B$ such that $E \succ C$ or (ii) there are such sets but for none of them it is true that $A \cup C \succsim B \cup C$. Case (i) can be ruled out by Lemma 2. If case (ii) holds, then, since $\succsim$ is complete, we must have $A \cup C \prec B \cup C$ for all sets C disjoint from $A \cup B$ such that $E \succ C$. So, $B \succsim_l A$ and the relation $\succsim_l$ is therefore complete. $\square$

Lemma 4 *Assume that $\succsim$ is an ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness and Averaging. Then, for all $A, B, C \in \mathcal{P}(X)$ such that $A \sim B \succ C$ and $C \cap (A \cup B) = \varnothing$,*

1. *$A \succ_l B$ if and only if $A \cup C \succ B \cup C$.*
2. *$A \sim_l B$ if and only if $A \cup C \sim B \cup C$.*

Proof. 1, $\Rightarrow$. $A \succ_l B$ implies $B \not\succsim_l A$. So, either there is no D disjoint from $A \cup B$ with $D \prec A$ (this is ruled out by Lemma 2) or $B \cup D \prec A \cup D, \forall D \prec A$. In particular, $A \cup C \succ B \cup C$.

1, $\Leftarrow$. Suppose $A \cup C \succ B \cup C$. This implies $A \succsim_l B$ (by definition of $\succsim_l$). Suppose $A \succ_l B$ does not hold. Since $\succsim_l$ is complete, $B \succsim_l A$ must hold so that, by definition of $\succsim_l$, there exists a set D such that $B \cup D \succsim A \cup D$, and $D \prec A$. But this contradicts balancedness. Hence $A \succ_l B$ must hold.

2, $\Rightarrow$. $A \sim_l B$ implies the existence of $D, D' \prec A$ such that $(D \cup D') \cap (A \cup B) = \varnothing$, $A \cup D \succsim B \cup D$ and $B \cup D' \succsim A \cup D'$. By balancedness, $A \cup C \succsim B \cup C$ and $B \cup C \succsim A \cup C$ and, so, $A \cup C \sim B \cup C$.

2, $\Leftarrow$ Obvious. $\square$

Lemma 5 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then, for every $E \in \mathcal{P}_*(X)$, there exists a mapping $p^E : \mathcal{P}^E(X) \to \mathbb{R}_{++}$ such that, for all*

$A, B \in \mathcal{P}^E(X)$, $A \succsim_l B$ iff $p^E(A) \geq p^E(B)$ and, for all disjoint $A, B \in \mathcal{P}^E(X)$, $p^E(A \cup B) = p^E(A) + p^E(B)$. Furthermore, p^E is unique up to a linear transformation.

Proof. Define the binary operation $\circ^E$ on $\mathcal{P}^E(X)$ as follows. If $A \cap B = \varnothing$, then $A \circ^E B = A \cup B$. Otherwise set $A \circ^E B = A' \cup B'$ for some $A', B' \in \mathcal{P}^E(X)$ such that $A' \cap B' = \varnothing$, $A' \cup C \sim A \cup C$ and $B' \cup D \sim B \cup D$ for some $C, D \prec E$ such that $(A \cup A') \cap C = \varnothing$ and $(B \cup B') \cap D = \varnothing$. The existence of such sets A', B' does not pose any difficulty, thanks to Richness. Indeed, by Lemma 2 and Averaging, there exists a set $C \in \mathcal{P}(X)$ such that $C \prec A \sim B$. By Averaging, $C \prec C \cup A \prec A$ and, using Richness, there exists a set A' such that $A' \cup C \sim A \cup C$, $A' \sim A$ and $A' \cap (C \cup A) = \varnothing$. Using an analogous reasoning, one can establish the existence of a set B' such that $B' \cup C \sim B \cup C$, $B' \sim B$ and $B' \cap (C \cup A \cup A') = \varnothing$.

Hence $\circ^E$ is defined for every pair $A, B \in \mathcal{P}^E(X)$, and the choice of the sets A', B' can be made by any rule whatsoever if there are several such sets for a given pair A, B. Finally we note that $\circ^E$ is closed in the set $\mathcal{P}^E(X)$ thanks to Averaging.

For any $E \in \mathcal{P}(X)$, we now show that the structure formed by the set $\mathcal{P}^E(X)$, the binary relation $\succsim_l$and the binary operation $\circ^E$ is what [3] (p. 73, definition 1) call a closed extensive measurement structure. That is to say, we establish that:

1. $\succsim_l$is a weak order: see Lemma 3;

2. $\circ^E$is weakly associative so that $A \circ^E (B \circ^E C) \sim_l (A \circ^E B) \circ^E C$ for every A, B and $C \in \mathcal{P}^E(X)$. The proof of this is obvious if A, B, C are mutually disjoint. Consider now the case where $A \cap B \cap C \neq \varnothing$. Let $A', B', C' \in \mathcal{P}(X)$ be mutually disjoint sets such that $A' \cup M \sim A \cup M$, $B' \cup N \sim B \cup N$, $C' \cup O \sim C \cup O$ for some $M, N, O \prec E$ with $(A \cup A') \cap M = (B \cup B') \cap N = (C \cup C') \cap O = \varnothing$. They exist thanks to Richness (the argument is similar to that employed in the definition of the binary operation $\circ^E$). We have $B \circ^E C = B' \cup C'$ and $A \circ^E (B \circ^E C) = A' \cup B' \cup C'$. We also have $A \circ^E B = A' \cup B'$ and $(A \circ^E B) \circ^E C = A' \cup B' \cup C'$, so that $A \circ^E (B \circ^E C) = (A \circ^E B) \circ^E C$. The reasoning is similar when some but not all pairwise intersections between A, B, C are not empty.

3. monotonicity holds (that is: $A \succsim_l B$ iff $A \circ^E C \succsim_l B \circ^E C$ iff $C \circ^E A \succsim_l C \circ^E B$). Since $\circ^E$ is obviously commutative, we just need to prove $A \succsim_l B$ iff $A \circ^E C \succsim_l B \circ^E C$. Choose A' and B' in $\mathcal{P}^E(X)$ such that $A' \cap C = \varnothing = B' \cap C$, $A' \cup D \sim A \cup D$ and $B' \cup D \sim B \cup D$ for some $D \prec E$ and disjoint from C. Thanks to Richness, this is always possible. Notice that $C \cup D \prec E$ by averaging. We have $A \succsim_l B$ iff $A \cup F \succsim B \cup F$ (by definition) iff $A' \cup D \succsim B' \cup D$ (by construction) iff $A' \cup C \cup D \succsim B' \cup C \cup D$ (by Balancedness and because $C \cup D \prec E$ thanks to Averaging) iff $A \circ^E C \succsim_l B \circ^E C$;

4. The Archimedean axiom:: if $A \succ_l B$, then, for any $C, D \in \mathcal{P}^E(X)$, there exists a positive integer n such that $nA \circ^E C \succsim_l nB \circ^E D$, where nA is defined inductively as: $1A = A$, $(n+1)A = nA \circ^E A$. It is immediate to see that this condition is implied by the Archimedean axiom.

By Theorem 1 of [3] (p.74), for any $E \in \mathcal{P}_*(X)$, there exists a mapping $p^E : \mathcal{P}^E(X) \to \mathbb{R}$ such that, for all $A, B \in \mathcal{P}^E(X)$, $A \succsim_l B$ iff $p^E(A) \geq p^E(B)$ and $p^E(A \circ^E B) = p^E(A) + p^E(B)$. Furthermore, p^E is unique up to a linear transformation.

We now show that $p^E(A) > 0$ for all $A \in \mathcal{P}^E(X)$. For any $A \in \mathcal{P}^E(X)$, we can find a set $B \in \mathcal{P}^E(X)$ such that $A \cap B = \varnothing$ (using Lemma 1). By definition of $\mathcal{P}_*(X)$, there is $D' \prec E$. By Lemma 1, there is $D \sim D' \prec E$ such that $D \cap (A \cup B) = \varnothing$. By Averaging, $D \prec B \cup D \prec B \sim A$. By Averaging again, $B \cup D \prec B \cup D \cup A \prec B$. By definition of $\succsim_l$, $B \prec_l A \cup B$. This implies $p^E(B) < p^E(A \cup B)$ and, since A and B are disjoint, $p^E(B) < p^E(A) + p^E(B)$ or, equivalently, $p^E(A) > 0$. □

MAKE SOME COMMENTS TO EXPLAIN THE RESULTS. The following lemma is quite close in spirit to lemma 10 in [1].

Lemma 6 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and The Archimedean axiom. If $E \in \mathcal{P}_*(X)$, there exists a mapping $\nu^E : \mathcal{P}(X) \to \mathbb{R}$ such that (i) $A \cap B = \varnothing$ implies $\nu^E(A \cup B) = \nu^E(A) + \nu^E(B)$ and (ii) $\nu^E(A) \geq 0$ iff $A \succsim E$ and $\nu^E(A) \leq 0$ iff $A \precsim E$.*

Proof. For a fixed $E \in \mathcal{P}_*(X)$, let $\mathcal{L} = \{a \in X : \{a\} \prec E\}$ and $\mathcal{U} = \{a \in X : \{a\} \succ E\}$. These sets are not empty (this is an almost immediate consequence of Lemma 2). Define M as an arbitrary set such that $E \prec M$.

We first define ν^E on $\mathcal{P}(\mathcal{L})$. Fix some $L \in \mathcal{L}$. By Richness, there is $U \in \mathcal{P}(\mathcal{U})$ such that $U \sim M$ and $U \cup L \sim E$. Set $\nu^E(L) = -p^M(U)$. By construction, $\nu^E(L)$ does not depend on the choice of U. Indeed, suppose there are several such U, say U and U'. Notice that $U \sim M \sim U'$, $U \cup L \sim E$ and $U' \cup L \sim E$. So, $U \cup L \sim U' \cup L$. Hence $U \sim_l U'$ and $p^M(U) = p^M(U')$.

Select $L_1, L_2 \in \mathcal{L}$, with $L_1 \cap L_2 = \varnothing$. By Averaging, $L_1 \cup L_2 \in \mathcal{L}$. Using Richness as above, we find two disjoint sets $U_1, U_2 \in \mathcal{P}(\mathcal{U})$ such that $U_1 \sim U_2 \sim M$, $U_1 \cup L_1 \sim E$ and $U_2 \cup L_2 \sim E$. By Averaging, $U_1 \cup U_2 \cup L_1 \cup L_2 \sim E$, $U_1 \cup U_2 \sim M$ and $U_2 \cup L_2 \sim E$. So,

$$\begin{aligned} \nu^E(L_1 \cup L_2) &= -p^M(U_1 \cup U_2) \\ &= -p^M(U_1) - p^M(U_2) \\ &= \nu^E(L_1) + \nu^E(L_2). \end{aligned} \tag{2}$$

This proves that ν^E is disjoint-additive over $\mathcal{L}$.

We now define ν^E on $\mathcal{P}(\mathcal{U})$. Take any $U \in \mathcal{P}(\mathcal{U})$. By Richness used in a similar (but this time "downward") way as above, there is $L \in \mathcal{P}(\mathcal{L})$ such that

$U \cup L \sim E$. Set $\nu^E(U) = -\nu^E(L)$. The mapping ν^E on $\mathcal{P}(\mathcal{U})$ does not depend on the choice of L. Indeed, suppose there are several such L, say L_1 and L_2 in $\mathcal{P}(\mathcal{L})$. We must prove that $\nu^E(L_1) = \nu^E(L_2)$. Suppose first $L_1 \cap L_2 = \emptyset$. Let $U_1, U_2 \in \mathcal{P}(\mathcal{U})$ be such that $U_1 \cap U = \emptyset = U_2 \cap U$, $U_1 \sim M \sim U_2$, $U_1 \cup L_1 \sim E \sim U_2 \cup L_2$. By Richness, such sets exist. We also have $U \cup L_1 \sim E \sim U \cup L_2$. By Averaging, $U_1 \cup L_1 \cup U \cup L_2 \sim E \sim U_2 \cup L_2 \cup U \cup L_1$. Hence, $U_1 \sim_l U_2$, $p^M(U_1) = p^M(U_2)$ and $\nu^E(L_1) = \nu^E(L_2)$. Suppose now $L_1 \cap L_2 \neq \emptyset$. By Richness used in the same way as above, there is $L_3 \in \mathcal{P}(\mathcal{L})$ such that $L_3 \cap (L_1 \cup L_2) = \emptyset$ and $U \cup L_3 \sim E$. Define U_3 by $U_3 \sim M$ and $U_3 \cup L_3 \sim E$. By richness, U_3 can be chosen disjoint from both U_1 and U_2. Since $U_1 \cup L_1 \sim U \cup L_3 \sim E \sim U_3 \cup L_3 \sim U \cup L_1$ and U, U_1 and U_3 are disjoint as are L_1 and L_2, it follows from Averaging that $U_1 \cup L_1 \cup U \cup L_3 \sim E \sim U_3 \cup L_3 \cup U \cup L_1$. Hence, $U_1 \sim_l U_3$ and, therefore, $p^M(U_1) = p^M(U_3)$. A similar reasoning can be performed for U_2 and U_3. We therefore have $p^M(U_1) = p^M(U_2) = p^M(U_3)$ and, as a result, $\nu^E(L_1) = \nu^E(L_3) = \nu^E(L_2)$.

The mapping ν^E on $\mathcal{P}(\mathcal{U})$ is additive. Indeed, consider two sets $U_1, U_2 \in \mathcal{P}(\mathcal{U})$, with $U_1 \cap U_2 = \emptyset$. Let us find two sets $L_1, L_2 \in \mathcal{P}(\mathcal{L})$ such that $U_1 \cup L_1 \sim E \sim U_2 \cup L_2$. Since the choice of L_1 and L_2 is not important, we can choose them disjoint (using Richness). By Averaging, $U_1 \cup U_2 \cup L_1 \cup L_2 \sim E$. So, $\nu^E(U_1 \cup U_2) = -\nu^E(L_1 \cup L_2) = -\nu^E(L_1) - \nu^E(L_2) = \nu^E(U_1) + \nu^E(U_2)$.

We define then ν^E on $\mathcal{P}(X)$. Take any $S \in \mathcal{P}(X)$. If $\{s\} \sim E$ for all $s \in S$, set $\nu^E(S) = 0$. Otherwise, we can express S as $S = L \cup U \cup R$ with $L = S \cap \mathcal{L}$, $U = S \cap \mathcal{U}$ and $R = S \setminus (L \cup U)$. By Averaging, $S \succsim E$ iff $L \cup U \succsim E$. Set $\nu^E(S) = \nu^E(L) + \nu^E(U)$. Disjoint-additivity is inherited from ν^E on $\mathcal{P}(\mathcal{U})$ and ν^E on $\mathcal{P}(\mathcal{L})$.

We must now check whether ν^E satisfies (ii). Suppose $S \succ E$. Then $(S \cap \mathcal{L}) \cup (S \cap \mathcal{U}) \succ E$. Using richness and averaging, one can find a superset L' of $S \cap \mathcal{L}$ belonging to $\mathcal{P}(\mathcal{L})$ such that $L' \cup (S \cap \mathcal{U}) \sim E$. As shown above, $-v^E(L'^E(S \cap \mathcal{U})$. Since $S \cap \mathcal{L} \subset L' \subseteq \mathcal{L}$, and , for every $L \in \mathcal{P}(\mathcal{L})$, $\nu^E(L) = -p^M(U) < 0$ for some set $U \in \mathcal{P}(\mathcal{U})$ we have that $0 > \nu^E(S \cap \mathcal{L}) > \nu^E(L')$ by disjoint-additivity. Now, by construction, $\nu^E(S) = \nu^E(S \cap \mathcal{L}) + \nu^E(S \cap \mathcal{U}) = \nu^E(S \cap \mathcal{L}) - \nu^E(L') > 0$.

Suppose now $S \prec E$. Then $(S \cap \mathcal{L}) \cup (S \cap \mathcal{U}) \prec E$. Using Averaging and Richness again, there is a superset U' of $S \cap \mathcal{U}$ belonging to $\mathcal{P}(\mathcal{U})$ such that $U' \cup (S \cap \mathcal{L}) \sim E$. By definition of the mapping ν^E, one has that $\nu^E(U'^E(S \cap \mathcal{L}) > 0$. Moreover, since $S \cap \mathcal{U} \subset U' \subseteq \mathcal{U}$ and $\nu^E(U) > 0$ for every $U \in \mathcal{P}(\mathcal{U})$, one has $\nu^E(U') > V^E(S \cap \mathcal{U}) > 0$ by disjoint-additivity. We have, by construction, $\nu^E(S) = \nu^E(S \cap \mathcal{L}) + \nu^E(S \cap \mathcal{U}) = \nu^E(S \cap \mathcal{U}) - \nu^E(U') < 0$.

Suppose finally $S \sim E$. Then $(S \cap \mathcal{L}) \cup (S \cap \mathcal{U}) \sim E$ so that $\nu^E(S \cap \mathcal{L}) = -\nu^E(S \cap \mathcal{U})$. We have, by construction, $\nu^E(S) = \nu^E(S \cap \mathcal{L}) + \nu^E(S \cap \mathcal{U}) = \nu^E(S \cap \mathcal{U}) - \nu^E(S \cap \mathcal{U}) = 0$. □

Notice that ν^E is defined only for $E \in \mathcal{P}_*(X)$, but it maps every set $A \in \mathcal{P}(X)$ on $\nu^E(A)$, even if A belongs to $m(X)$ or $M(X)$. We now prove a lemma that is quite similar in spirit to lemma 12 in [1].

Lemma 7 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Then the family $\{\nu^E : E \in \mathcal{P}_*(X)\}$ is spanned by any two of its members ν^A and ν^B (with ν^A and ν^B linearly independent). That is, for any mapping ν^E in the family, there are two real numbers α^E, β^E such that $\nu^E = \alpha^E \nu^A + \beta^E \nu^B$.*

Proof. Take any $A, B \in \mathcal{P}_*(X)$ such that $A \succ B$. We first show that the mapping ν^B numerically represents the probability ordering $\succsim^l$ on $\mathcal{P}^A(X)$. That is to say we first establish that

$$\nu^B(S) \geq \nu^B(T) \iff S \cup F \succsim T \cup F. \tag{3}$$

holds for every S and T such that $S \sim T \sim A$ and every $F \prec A$. Consider indeed sets S and T with $A \sim S \sim T$. By lemma 6, $\nu^A(S) = 0 = \nu^A(T)$. By construction, $\nu^B(S) > 0$. By richness and unboundedness, there is L_1 such that $L_1 \cap (S \cup T) = \varnothing$ and $S \cup L_1 \sim B$. By Averaging $L_1 \prec B$. By Lemma 6, $\nu^B(S) + \nu^B(L_1) = \nu^B(S \cup L_1) = 0$. Suppose $\nu^B(T) \geq \nu^B(S)$. Then, $\nu^B(T \cup L_1) = \nu^B(T) + \nu^B(L_1) \geq 0$. By Lemma 6, $T \cup L_1 \succsim B \sim S \cup L_1$. By Balancedness, $T \cup F \succsim S \cup F$ for any $F : F \prec A, F \cap (S \cup T) = \varnothing$. A similar argument shows that $\nu^B(T) > \nu^B(S) \Rightarrow T \cup F \succ S \cup F$ for any $F : F \prec A, F \cap (S \cup T) = \varnothing$.

Conversely, suppose $T \cup F \succsim S \cup F$ for some $F : F \prec A, F \cap (S \cup T) = \varnothing$. By Richness, there is L_2 such that $L_2 \cap (S \cup T) = \varnothing$, $S \cup L_2 \sim B$. By Averaging, $L_2 \prec B$. By Balancedness, $T \cup L_2 \succsim S \cup L_2 \sim B$. By Lemma 6, $\nu^B(T) + \nu^B(L_2) \geq 0$. Since $\nu^B(S) + \nu^B(L_2) = 0$, we obtain $\nu^B(T) \geq \nu^B(S)$. The same argument holds if we suppose $T \cup F \succ S \cup F$, and this establishes (3).

Choose now $S, T \in \mathcal{P}^A(X)$ and $B, D \prec A$ with $D \cap (S \cup T) = \varnothing$. By Richness, this is possible. Suppose without loss of generality that $S \cup D \precsim T \cup D$. By iterative application of Richness, there exist sets $S_1, S_2, ...$ such that, for every $i \neq j \in \mathbb{N}$, $S \cap (\bigcup_{i \in \mathbb{N}} S_i) = \varnothing = S_i \cap S_j = S_i \cap D$, $S_i \sim S$ and $\nu^B(S) = \nu^B(S_i)$. Similarly, there exist $T_1, T_2, ...$ such that, for every $i \neq j \in \mathbb{N}$, $T \cap (\bigcup_{i \in \mathbb{N}} T_i) = \varnothing = T_i \cap T_j = T_i \cap D$, $T_i \sim T$ and $\nu^B(T) = \nu^B(T_i)$.

For every positive integer n, there is a largest integer $q(n)$ such that $\bigcup_{i=1}^{q(n)} S_i \cup D \precsim \bigcup_{i=1}^{n} T_i \cup D$ because $\nu^B(\bigcup_{i=1}^{p} S_i) = p\nu^B(S_i)$ and is therefore unbounded when p increases. Notice that $q(n) \geq n$ because $\nu^B(S) \leq \nu^B(T)$. We thus have $\bigcup_{i=1}^{q(n)} S_i \cup D \precsim \bigcup_{i=1}^{n} T_i \cup D \prec \bigcup_{i=1}^{q(n)+1} S_i \cup D$, for every positive integer n. Since the sets $\bigcup_{i=1}^{q(n)} S_i$, $\bigcup_{i=1}^{n} T_i$ and $\bigcup_{i=1}^{q(n)+1} S_i$ are all equivalent to A (by Averaging) and thanks to (3), we have $\nu^B(\bigcup_{i=1}^{q(n)} S_i) \leq \nu^B(\bigcup_{i=1}^{n} T_i) < \nu^B(\bigcup_{i=1}^{q(n)+1} S_i)$. The mapping ν^B being additive, we may write $q(n)\nu^B(S) \leq n\nu^B(T) < (q(n) + 1)\nu^B(S)$ and

$$\frac{q(n)}{n}\ \nu^B(S) \leq \nu^B(T) < \frac{q(n)+1}{n}\ \nu^B(S),\ \forall n \in \mathbb{N}_0$$

so that $\nu^B(T) = \lim_{n \to \infty} \frac{q(n)}{n}\ \nu^B(S)$. Following the same reasoning with any $C \in \mathcal{P}_*(X)$ with $C \prec A$ instead of B yields $\nu^C(T) = \lim_{n \to \infty} \frac{q(n)}{n}\ \nu^C(S)$. So,

$\nu^B(T)/\nu^B(S) = \nu^C(T)/\nu^C(S)$. Since this holds for any $S, T \sim A$, this proves that $\nu^B(S) = k\nu^C(S)$ for some positive constant k and for all S such that $\nu^A(S) = 0$.

Define $\nu_{ABC}(S) = (\nu^A(S), \nu^B(S), \nu^C(S))$ for all $S \in \mathcal{P}(X)\}$. Then $\{x \in \mathbb{R}^3 : x_1 = 0\} \cap \nu_{ABC}(\mathcal{P}(X))$ is contained in the ray $\{(0, kt, t) : t \geq 0\}$. Since $A \in \mathcal{P}_*(X)$, there is S such that $S \succ A$ or $S \prec A$, whence the set $\{x \in \nu_{ABC}(\mathcal{P}(X)) : x_1 \neq 0\}$ is not empty. We can therefore select vectors $x^0, x^1 \in \nu_{ABC}(\mathcal{P}(X))$ such that $x_1^0 = 0$ and $x_1^1 \neq 0$. Let S^0 and S^1 be such that $\nu_{ABC}(S^0) = x^0$ and $\nu_{ABC}(S^1) = x^1$.

We show that these two vectors, together, span $\nu_{ABC}(\mathcal{P}(X))$. Let $x \in \nu_{ABC}(\mathcal{P}(X))$, with $\nu_{ABC}(S) = x$. We proceed by cases, assuming $x_1^1 > 0$ (the case $x_1^1 < 0$ being symmetric).

1. Suppose $x_1 = 0$. Since $\{x \in \mathbb{R}^3 : x_1 = 0\} \cap \nu_{ABC}(\mathcal{P}(X))$ is contained in the ray $\{(0, kt, t) : t \geq 0\}$, we have $x = kx^0$.

2. Suppose $x_1 > 0$. By Richness, there is $T : T \cup S^1 \sim S^0$. Hence, $\nu^A(T) = -\nu^A(S^1)$. By Richness, there is $R : R \cup T \sim S^0, R \sim S$. Hence, $\nu^A(R) = \nu^A(S^1)$. Since $R \sim S \succ A, B, C$, we know that $\nu^C(R) = \alpha\nu^A(R)$ and $\nu^C(S) = \alpha\nu^A(S)$ for some $\alpha \in \mathbb{R}$. For the same reason, $\nu^B(R) = \beta\nu^A(R)$ and $\nu^B(S) = \beta\nu^A(S)$ for some $\beta \in \mathbb{R}$. So, $\nu^C(R)/\nu^C(S) = \nu^A(R)/\nu^A(S)$ and $\nu^B(R)/\nu^B(S) = \nu^A(R)/\nu^A(S)$. In other words, $\nu_{ABC}(R)$ and $\nu_{ABC}(S)$ are in the same ray and $\nu_{ABC}(S) = \gamma\nu_{ABC}(R)$ for some $\gamma \in \mathbb{R}$.

   Since $T \cup S^1 \sim S^0$, we know that $\nu_{ABC}(T \cup S^1)$ is in the same ray as x^0. So, $\nu_{ABC}(T \cup S^1) = \nu_{ABC}(T) + x^1 = \lambda x^0$ for some $\lambda > 0$. Similarly, since $T \cup R \sim S^0$, we know that $\nu_{ABC}(T \cup R)$ is in the same ray as x^0. So, $\nu_{ABC}(T \cup R) = \nu_{ABC}(T) + \nu_{ABC}(R) = \lambda x^0 - x^1 + \nu_{ABC}(R) = \lambda'^0$ for some $\lambda' > 0$. Whence $\nu_{ABC}(R) = \lambda'^0 - \lambda x^0 + x^1$. We can therefore write $\nu_{ABC}(S) = \gamma(\lambda'^0 - \lambda x^0 + x^1)$. This proves that x is spanned by x^0 and x^1.

3. Suppose $x_1 < 0$. By Richness, there is $T : T \cup S \sim S^0$ and, hence, $\nu_{ABC}(T \cup S)$ is in the same ray as x^0. So, $\nu_{ABC}(T \cup S) = \nu_{ABC}(T) + x = \lambda x^0$ for some $\lambda > 0$. So, $x = \lambda x^0 - \nu_{ABC}(T)$. Put another way, x is spanned by x^0 and $\nu_{ABC}(T)$. We have seen in case 2 that $\nu_{ABC}(T)$ is spanned by x^0 and x^1. So, actually, x is spanned by x^0 and x^1.

So, there are two real numbers λ, γ such that, for any $S \in \mathcal{P}(X)$,

$$\nu_{ABC}(S) = \lambda\nu_{ABC}(S^0) + \gamma\nu_{ABC}(S^1). \tag{4}$$

In particular, $\nu^A(S) = \lambda\nu^A(S^0) + \gamma\nu^A(S^1) = \gamma\nu^A(S^1)$ because $\nu^A(S^0) = 0$. So, $\gamma = \nu^A(S)/\nu^A(S^1)$. From (4), we also derive $\nu^C(S) = \lambda\nu^C(S^0) + \gamma\nu^C(S^1)$ which yields $\lambda = (\nu^C(S) - \gamma\nu^C(S^1))/\nu^C(S^0)$. From (4), we finally derive $\nu^B(S) = \lambda\nu^B(S^0) + \gamma\nu^B(S^1)$. Let us substitute λ and γ in this equation.

We obtain

$$\nu^B(S) = \frac{\nu^C(S) - (\nu^A(S)/\nu^A(S^1))\nu^C(S^1)}{\nu^C(S^0)}\nu^B(S^0) + \nu^A(S)\nu^B(S^1)/\nu^A(S^1),$$

showing that ν^B is a linear combination of ν^A and ν^C.

So, for every $A, B, C \in \mathcal{P}_*(X)$, such that none of them are indifferent, there are two real numbers α, β such that $\nu^A = \alpha\nu^B + \beta\nu^C$. Consider now A, B, C such that $B \not\sim C$. We select D, D' not indifferent to any of A, B, C and such that $D \not\sim D'$. We can express each of ν^A, ν^B, ν^C as a linear combination of ν^D and $\nu^{D'}$. For instance,

$$\nu^A = \alpha_A\nu^D + \beta_A\nu^{D'}, \tag{5}$$

$$\nu^B = \alpha_B\nu^D + \beta_B\nu^{D'} \tag{6}$$

$$\nu^C = \alpha_C\nu^D + \beta_C\nu^{D'}. \tag{7}$$

From (6) and (7), we derive

$$\nu^D = \frac{\nu^C\beta_B - \nu^B\beta_C}{\alpha_C\beta_B - \alpha_B\beta_C}$$

and

$$\nu_{D'} = \frac{\nu^C\alpha_B - \nu^B\alpha_C}{\beta_C\alpha_B - \beta_B\alpha_C}.$$

We substitute ν^D and $\nu^{D'}$ in (5) and we obtain that ν^A is a linear combination of ν^B and ν^C. This suffices to show the entire space $\{\nu^E : E \in \mathcal{P}_*(X)\}$ can be spanned by any two of its members ν^B, ν^C with $B \not\sim C$, since the selection of A, B, C in the proof was arbitrary. $\square$

Define $\mathrm{span}(\nu^A, \nu^B)$ as the set of all possible linear combinations of ν^A and ν^B. In light of Lemma 7, $\mathrm{span}(\nu^A, \nu^B) = \mathrm{span}(\nu^C, \nu^D)$ for all $A, B, C, D \in \mathcal{P}_*(X)$ such that ν^A and ν^B (resp. ν^C and ν^D) are linearly independent. It therefore makes sense to define $\mathcal{S} = \mathrm{span}(\nu^A, \nu^B)$ for some $A, B \in \mathcal{P}_*(X)$ with ν^A and ν^B linearly independent.

Lemma 8 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Let A, B, C be three sets in $\mathcal{P}_*(X)$. It is impossible to have $-\lambda\nu^A = \delta\nu^B + (1-\delta)\nu^C$ for some $\lambda \in \mathbb{R}_{++}$ and $\delta \in [0, 1]$.*

Proof. We consider two cases.

(1) $\nu^B = k\nu^C$ for some $k \in \mathbb{R}_{++}$. Then $\nu^A = -\lambda\nu^B$. This is not possible because, by Lemma 6, for any $D \prec B$, we have $\nu^B(D) < 0$ and $\nu^A(D) < 0$. The cases $\nu^A = k\nu^C$ and $\nu^A = k\nu^B$ are treated in the same way.

(2) $A \succ B \succ C$ (the 5 other orderings are treated in the same way) .

By Lemma 7, ν^A and ν^B span $\{\nu^E : E \in \mathcal{P}_*(X)\}$. For every $C \in \mathcal{P}_*(X)$, let $\alpha(C)$ and $\beta(C)$ solve $\nu^C = \alpha(C)\nu^A + \beta(C)\nu^B$. Since $C \prec B$, we must have

$\alpha(C) < 0$ or $\beta(C) < 0$, otherwise, for any $S : C \prec S \prec B$, it is impossible to have $\nu^C(S) > 0$. Simultaneously, we must also have $\beta(C) > 0$ because $\nu^C(A)$ must be positive. So, we must have $\alpha(C) < 0 < \beta(C)$. Assume for contradiction that $-\lambda\nu^A = \delta\nu^B + (1-\delta)\nu^C$ for some $\lambda \in \mathbb{R}_{++}$ and $\delta \in]0,1[$. This implies

$$\nu^C = \frac{\lambda}{\delta - 1}\nu^A + \frac{\delta}{\delta - 1}\nu^B$$

with $\delta/(\delta - 1) < 0$. This is in contradiction with $0 < \beta(C)$. $\square$

Lemma 9 *Let us assume Weak Order, Balancedness, Averaging, non-triviality Richness and the Archimedean axiom. There exist then a disjoint-additive mapping $\mu : \mathcal{P}(X) \to \mathbb{R}$ such that $\mu(C) > 0$ for all $C \in \mathcal{P}_*(X)$ and $\mu \in \mathcal{S}$.*

Proof. Let us choose some $B, C \in \mathcal{P}_*(X)$ with $C \prec B$. By Lemma 7, ν^B and ν^C span $\{\nu^E : E \in \mathcal{P}_*(X)\}$. For every $A \in \mathcal{P}_*(X)$, let $\alpha(A)$ and $\beta(A)$ solve $\nu^A = \alpha(A)\nu^B + \beta(A)\nu^C$. If $A \prec B, C$, then we must have $\alpha(A) < 0$ or $\beta(A) < 0$, otherwise, for any $S : A \prec S \prec B, C$, it is impossible to have $\nu^A(S) > 0$. Simultaneously, we must also have $\beta(A) > 0$ because $\nu^A(B)$ must be positive. So, we must have $\alpha(A) < 0 < \beta(A)$. Define $\rho(A) = -\beta(A)/\alpha(A)$. We have $\rho(A) > 0$ for all $A \in \mathcal{P}_*(X)$ with $A \prec C$.

If $A' \in \mathcal{P}_*(X)$ and $A' \prec A \prec C$, then $\rho(A') < \rho(A)$. Suppose, on the contrary, $\rho(A') \geq \rho(A)$. Since $\nu^A(A) = \alpha(A)\nu^B(A) + \beta(A)\nu^C(A) = 0$, we have

$$-\frac{\beta(A)}{\alpha(A)} = \rho(A) = \frac{\nu^B(A)}{\nu^C(A)} \leq -\frac{\beta(A')}{\alpha(A')} = \rho(A').$$

Hence $\nu^B(A)\alpha(A'^C(A)\beta(A')$ and $\nu^{A'}(A) = \nu^B(A)\alpha(A'^C(A)\beta(A') \leq 0$, which implies $A \prec A'$. A contradiction. Notice that the converse is also true. So, for all $A, A' \in \mathcal{P}_*(X)$ with $A, A' \prec C$, $A' \precsim A$ iff $\rho(A') \leq \rho(A)$.

Similarly, it is easy to prove that, for all $A, A' \in \mathcal{P}_*(X)$ with $A, A' \succ B$, we have $\rho(A) > 0$ and $A' \precsim A$ iff $\rho(A') \leq \rho(A)$.

Define $Q = \{\rho(A) : A \in \mathcal{P}_*(X), A \prec C\}$. This set has a greatest lower bound $\rho^* \geq 0$ (because we have proved that $\rho(A) > 0$ for all $A \in \mathcal{P}(X) : A \prec C$). Actually, $\rho^* > 0$. Indeed, assume for contradiction that $\rho^* = 0$. Since $B \in \mathcal{P}_*(X)$, there is $D \in \mathcal{P}_*(X)$ such that $D \succ B$. Because $\rho^* = 0$, there is $F \in \mathcal{P}_*(X)$ with $\rho(F)$ close to zero and such that ν^D, ν^B and ν^F are as ν^A, ν^B and ν^C in Lemma 8. Yet, this is not possible. So, $\rho^* > 0$.

Furthermore $\rho^* \notin Q$ because the set $\{A \in \mathcal{P}_*(X) : A \prec C\}$ has no minimal element. Define μ as one of the elements in the ray $\{x(-\nu^B + \rho^*\nu^C) : x > 0\}$. For instance, define $\mu = -\nu^B + \rho^*\nu^C$. By construction, $\mu \in \mathrm{span}(\nu^B, \nu^C)$.

Is the next paragraph interesting ? It does not show that μ is unique, but it shows instead that, if we use a specific technique to construct μ, then μ does not depend on the choice of B and C.

We now show that the ray containing μ is independent of the choice of B and C. Suppose first that we use B' instead of B, with $C \prec B'$. For every $A \in \mathcal{P}_*(X)$, let $\gamma(A)$ and $\delta(A)$ solve $\nu^A = \gamma(A)\nu^{B'} + \delta(A)\nu^C$. Define $\sigma(A) = -\delta(A)/\gamma(A)$. Define σ^* as the greatest lower bound of the set $\{\sigma(A) : A \in \mathcal{P}_*(X), A \prec C\}$ and $\mu'^{B'} + \sigma^*\nu^C$. We want to show that μ' constructed using B' and C belongs to the same ray as μ. By Lemma 7, we can write $\nu^{B'} = p\nu^B + q\nu^C$ for some $p, q \in \mathbb{R}$. We have $\nu^{B'}(C) = p\nu^B(C) + q\nu^C(C) = p\nu^B(C)$. Since $\nu^{B'}(C) < 0$ and $\nu^B(C) < 0$, we find $p > 0$. We have $\nu^A = \gamma(A)\nu^{B'} + \delta(A)\nu^C = \gamma(A)(p\nu^B + q\nu^C) + \delta(A)\nu^C$. In particular, $\nu^A(A) = 0 = \gamma(A)(p\nu^B(A) + q\nu^C(A)) + \delta(A)\nu^C(A)$. Hence, $\sigma(A) = -\delta(A)/\gamma(A) = (p\nu^B(A) + q\nu^C(A))/\nu^C(A) = p(\nu^B(A)/\nu^C(A)) + q$. Notice that $\nu^A(A) = 0$ also implies $\nu^B(A)/\nu^C(A) = -\beta(A)/\alpha(A) = \rho(A)$. So, $\sigma(A) = p\rho(A) + q$. This yields $\sigma^* = p\rho^* + q$. We now rewrite μ' as $-\nu^{B'} + \sigma^*\nu^C = -p\nu^B - q\nu^C + \sigma^*\nu^C = -p\nu^B - q\nu^C + (p\rho^* + q)\nu^C = p(-\nu^B + \rho^*\nu^C) = p\mu$. This shows that the ray containing μ does not depend on B as long as $C \prec B$. A similar reasoning shows that the ray containing μ does not depend on C as long as $C \prec B$. In conclusion, μ is unique up to a multiplicative constant.

We now prove that $\mu(A) > 0$ for all $A \in \mathcal{P}_*(X)$. Suppose, on the contrary, $\mu(A) \leq 0$ for some $A \in \mathcal{P}_*(X)$. By definition of $\mathcal{P}_*(X)$, there are $B, C : A \prec C \prec B$. We can write $\mu^*(A) = -\nu^B(A) + \rho^*\nu^C(A) \leq 0$. So, $\rho^*\nu^C(A) \leq \nu^B(A)$ and $\rho^* \geq \nu^B(A)/\nu^C(A)$ because $\nu^C(A) < 0$. Since $\nu^A(A) = \alpha(A)\nu^B(A) + \beta(A)\nu^C(A) = 0$, we have $\rho^* \geq -\beta(A)/\alpha(A)$. This is impossible because $\rho^* \notin Q$. This contradiction proves that $\mu(A) > 0$ for all $A \in \mathcal{P}_*(X)$.

Do we need next paragraph ?

We now show that $\alpha(A) > 0$ for all $A \in \mathcal{P}_*(X)$. Suppose, on the contrary, there is $A : \alpha(A) \leq 0$. We easily find that $\alpha(A) < 0$ because $\alpha(A) = 0$ would yield $\nu^A(A) = 0 = \beta(A)\mu(A)$. But $\mu(A) > 0$ and $\beta(A) \neq 0$ (otherwise ν^A is identically zero). So, it is not possible that $\nu^A(A) = 0$ and, hence, $\alpha(A) \neq 0$. By unboundedness, there is $D \succ C, A$. So, $\nu^A(D) = \alpha(A)\nu^C(D) + \beta(A)\mu(D) > 0$. Since $\alpha(A) < 0$, $\nu^C(D) > 0$ and $\mu(D) > 0$ we conclude $\beta(A) > 0$. We also have $\nu^A(C) = \beta(A)\mu(C)$. Since $\beta(A) > 0$ and $\mu(C) > 0$, we find $\nu^A(C) > 0$ or, in other words, $C \succ A$. Let us now compute $\rho(A) = -\beta(A)/\alpha(A) = \nu^C(A)/\mu(A)$. Since $C \succ A$, we find $\nu^C(A) < 0$ and, hence, $\rho(A) < 0$. But we have previously seen that $\rho(E) > 0$ for every $E \prec C$. In particular, for A. This contradiction proves that $\alpha(A) > 0$ for all $A \in \mathcal{P}_*(X)$.

The mapping μ is additive for disjoint sets because it is the linear combination of two disjoint-additive mappings. It clearly belongs to $\mathcal{S}$ because it is the linear combination of two independent elements of $\mathcal{S}$. $\square$

Lemma 10 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Choose any $C \in \mathcal{P}_*(X)$ and define $\nu = \nu^C$. Then, for all $A, B \in \mathcal{P}_*(X)$, $\nu(A)/\mu(A) \geq \nu(B)/\mu(B)$ iff $A \succsim B$.*

Proof. For every $A \in \mathcal{P}_*(X)$, let $\alpha(A)$ and $\beta(A)$ solve $\nu^A = \alpha(A)\nu + \beta(A)\mu$. Such $\alpha(A)$ and $\beta(A)$ necessarily exist because ν and μ belong to $\mathcal{S}$ and are linearly independent.

By construction, $\nu^A(A) = 0 = \alpha(A)\nu(A) + \beta(A)\mu(A)$ or, equivalently, $\frac{\nu(A)}{\mu(A)} = \frac{-\beta(A)}{\alpha(A)}$. So, in order to show that ν/μ is a numerical representation of $\succsim$ on $\mathcal{P}_*(X)$, it suffices to show that $-\beta/\alpha$ represents $\succsim$ on $\mathcal{P}_*(X)$. Notice first that $-\beta/\alpha$ is well-defined because $-\beta(A)/\alpha(A) = \nu(A)/\mu(A)$ and $\mu(A) > 0$ for all $A \in \mathcal{P}_*(X)$. Pick any $A, B \in \mathcal{P}_*(X)$ with $A \succsim B$. By construction, $\nu^B(A) \geq 0$. From this, we derive $\alpha(B)\nu(A) + \beta(B)\mu(A) \geq 0$ and $\nu(A) \geq -\beta(B)\mu(A)/\alpha(B)$. We also have $\alpha(A)\nu(A) + \beta(A)\mu(A) = 0$ or, equivalently, $\nu(A) = -\beta(A)\mu(A)/\alpha(A)$. So, $-\beta(A)\mu(A)/\alpha(A) \geq -\beta(B)\mu(A)/\alpha(B)$ or, after simplification, $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. So, we have proved that $A \succsim B$ implies $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. Proving the converse is done following the inverse path. This concludes the proof that ν/μ is a numerical representation of $\succsim$ on $\mathcal{P}_*(X)$. □

Lemma 11 *Let us assume Weak Order, Balancedness, Averaging, non-triviality Richness and the Archimedean axiom. Then, for all $A, B \in \mathcal{P}_*(X)$, $A \sim B$ and $A \sim_l B$ implies $\nu(A) = \nu(B)$ and $\mu(A) = \mu(B)$.*

Proof. $A \sim_l B$ implies $A \cup C \sim B \cup C$ for some $C \prec A$. So, $\nu(A)/\mu(A) = \nu(B)/\mu(B)$ and

$$\frac{\nu(A \cup C)}{\mu(A \cup C)} = \frac{\nu(B \cup C)}{\mu(B \cup C)}.$$

Using the disjoint-additivity of ν and μ,

$$\frac{\nu(A) + \nu(C)}{\mu(A) + \mu(C)} = \frac{\nu(B) + \nu(C)}{\mu(B) + \nu(C)}.$$

Let us replace $\nu(A)$ in this equation by $\nu(B)\mu(A)/\mu(B)$. We obtain

$$\frac{\nu(B)\mu(A)/\mu(B) + \nu(C)}{\mu(A) + \mu(C)} = \frac{\nu(B) + \nu(C)}{\mu(B) + \nu(C)}$$

or,

$$\begin{aligned} &\nu(B)\mu(A)\mu(B) + \nu(B)\mu(A)\mu(C) + \nu(C)\mu(B)\mu(B) + \nu(C)\mu(B)\mu(C) \\ &\quad = \nu(B)\mu(A)\mu(B) + \nu(B)\mu(B)\mu(C) + \nu(C)\mu(A)\mu(B) + \nu(C)\mu(B)\mu(C). \end{aligned}$$

After some simplifications and reordering some terms, we find

$$\nu(B)\mu(C)(\mu(A) - \mu(B)) = \nu(C)\mu(B)(\mu(A) - \mu(B)).$$

If $\mu(A) - \mu(B) = 0$, then $\nu(B)/\mu(B) = \nu(C)/\mu(C)$ and $B \sim C$, which is impossible. So we conclude that $\mu(A) - \mu(B) \neq 0$ and, hence, $\mu(A) = \mu(B)$ and $\nu(A) = \nu(B)$. □

Lemma 12 *Let us assume Weak Order, Balancedness, Averaging, non-triviality Richness and the Archimedean axiom. Then, for any $C \in \mathcal{P}_*(X)$ and $\epsilon > 0$, there is $D \sim C$ such that $\mu(D) < \epsilon$.*

Proof. Because $C \in \mathcal{P}_*(X)$, there are $A, B \in \mathcal{P}_*(X)$ such that $C \succ A \succ B$. By Lemma 1, there is a set A' such that $A' \sim A$ and $A' \cap B = \varnothing$. By Averaging, $A \succ A' \cup B \succ B$. By Richness, there are sets $A_1, A_2, \ldots$ such that, for $i \in \{1, 2, \ldots\}$, $A \sim A_i$, $A_i \cap (A \bigcup_{j=1}^{i-1} A_j) = \varnothing$ and $A_i \cup B \sim A' \cup B$. By Lemma 11, $\nu(A_i) = \nu(A)$ and $\mu(A_i) = \mu(A)$ for $i \in \{1, 2, \ldots\}$. Some of the sets $A_1, A_2, \ldots$ may intersect with C, but the number of such sets is necessarily finite. So, if we drop them, we still have an infinite series of sets $A_1, A_2, \ldots$ We therefore assume hereafter that $A_i \cap C = \varnothing$ for $i \in \{1, 2, \ldots\}$.

By Averaging, $C \succ C \bigcup_{j=1}^{k} A_j \succ A$, for any $k \in \{1, 2, \ldots\}$. By Richness, for any $k \in \{1, 2, \ldots\}$, there is D such that $D \cap (A \cup C) = \varnothing$, $D \sim C$ and $D \cup A \sim C \bigcup_{j=1}^{k} A_j$. By Lemma 10, $\nu(C)/\mu(C) = \nu(D)/\mu(D)$ and, for all $i \in \{1, 2, \ldots\}$, $\nu(A)/\mu(A) = \nu(A_i)/\mu(A_i)$ and

$$\frac{\nu(D \cup A)}{\mu(D \cup A)} = \frac{\nu(C \bigcup_{j=1}^{i} A_j)}{\mu(C \bigcup_{j=1}^{i} A_j)}.$$

Using the disjoint-additivity of ν and μ,

$$\frac{\nu(D) + \nu(A)}{\mu(D) + \mu(A)} = \frac{\nu(C) + \sum_{j=1}^{i} \nu(A_j)}{\mu(C) + \sum_{j=1}^{i} \mu(A_j)} = \frac{\nu(C) + k\nu(A)}{\mu(C) + k\mu(A)}.$$

Hence

$$(\nu(D) + \nu(A))\ (\mu(C) + k\mu(A)) = (\nu(C) + k\nu(A))\ (\mu(D) + \mu(A)).$$

If we replace in this equation $\nu(D)$ by $\nu(C)\mu(D)/\mu(C)$ and perform some elementary algebra (the same as in Lemma 11), we obtain

$$\nu(A)\mu(C)(\mu(C) - k\mu(D)) = \nu(C)\mu(A)(\mu(C) - k\mu(D)).$$

If $\mu(C) - k\mu(D) = 0$, then $\nu(A)/\mu(A) = \nu(C)/\mu(C)$ and $a \sim C$, which is impossible. We therefore conclude that $\mu(C) - k\mu(D) \neq 0$ and, hence, $\mu(D) = \mu(C)/k$. If we choose k large enough, we can thus guarantee $\mu(D) < \epsilon$. □

Lemma 13 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. There exist then a disjoint-additive mapping $\mu_+ : \mathcal{P}(X) \to \mathbb{R}$ such that $\mu_+(S) > 0$ for all $S \in \mathcal{P}(X)$ and $\mu_+ \in \mathcal{S}$.*

Proof. If $\mu(S) > 0$ for all $S \in \mathcal{P}(X)$, we define $\mu_+ = \mu$ and the proof is done.

Otherwise, let us first prove that $\mu(S) \geq 0$ for all $S \in m(X)$. Assume for contradiction that $S \in m(X)$ and $\mu(S) < 0$ and choose a set $T \in \mathcal{P}_*(X)$ with $S \cap T = \varnothing$, $\nu(T) < 0$ and $\mu(T)$ very small (thanks to Lemma 12). Consider the set $D = T \cup S$. Its numerical representation is

$$\frac{\nu(T) + \nu(S)}{\mu(T) + \mu(S)}.$$

The numerator is negative. If $\mu(T)$ is small enough, we are sure that the denominator is also negative. Hence $\nu(D)/\mu(D) > 0$ and $D \succ T$. Yet this is not possible because, by Averaging, $T \succ D$.

Using a similar argument, we can prove that $\mu(S) \geq 0$ for all $S \in M(X)$.

We now claim that it is not possible to have $\mu(S) = 0 = \mu(T)$ for some $S \in m(X)$, $T \in M(X)$. Assume for contradiction that $\mu(S) = 0 = \mu(T)$ for some $S \in m(X)$, $T \in M(X)$. By Averaging, $S \cup T \in \mathcal{P}_*(X)$. This implies $\mu(S \cup T) > 0$. But, using the disjoint-additivity of μ, we find that $\mu(S \cup T) = 0$ although $S \cup T \in \mathcal{P}_*(X)$. This contradiction concludes the proof of the claim.

Suppose now that $\mu(S) = 0$ for some $S \in m(X)$. This implies $\mu(T) > 0$ for all $T \in \mathcal{P}_*(X) \cup M(X)$. We know from Lemma 9 that $\mu = -\nu^B + \rho^*\nu^C$ for some $B, C \in \mathcal{P}_*(X)$. If we choose $\rho_+ < \rho^*$ and we define $\mu_+ = -\nu^B + \rho_+\nu^C$, we are sure that $\mu(S) > 0$. If, in addition, we choose ρ_+ very close to ρ^*, we can guarantee that $\mu(T) > 0$ for all $T \in \mathcal{P}_*(X)$. The mapping μ_+ is clearly disjoint-additive and it belongs to $\mathcal{S}$. We still have to prove that $\mu_+(T) > 0$ for all $T \in m(X)$. If $T \neq S$ and $\mu(T) > 0$, then the proof is obvious because we have chosen ρ_+ very close to ρ^*. If $T \neq S$ and $\mu(T) = 0$, the proof is not difficult. Remember that $\mu(T) = -\nu^B(T) + \rho^*\nu^C(T)$, where $\nu^B(T) < 0$ and $\nu^C(T) < 0$. So, if we choose $\rho_+ < \rho^*$, then $\mu_+(T) = -\nu^B(T) + \rho_+\nu^C(T)$ is necessarily larger that $\mu(T)$ and, hence, positive.

The case where $\mu(S) = 0$ for some $S \in M(X)$ is handled in the same way. $\square$

In the proof of Lemma 13, it is clear that μ_+ is not unique!

We now need to prove and equivalent of Lemma 10 with μ_+ instead of μ.

Lemma 14 *Assume that $\succsim$ is a non-trivial ordering on $\mathcal{P}(X)$ satisfying Richness, Balancedness, Averaging and the Archimedean axiom. Choose any $C \in \mathcal{P}_*(X)$ and define $\nu = \nu^C$. Then, for all $A, B \in \mathcal{P}_*(X)$, $\nu(A)/\mu_+(A) \geq \nu(B)/\mu_+(B)$ iff $A \succsim B$.*

Proof. For every $A \in \mathcal{P}_*(X)$, let $\alpha(A)$ and $\beta(A)$ solve $\nu^A = \alpha(A)\nu + \beta(A)\mu_+$. Such $\alpha(A)$ and $\beta(A)$ necessarily exist because ν and μ_+ belong to $\mathcal{S}$ and are linearly independent.

By construction, $\nu^A(A) = 0 = \alpha(A)\nu(A) + \beta(A)\mu_+(A)$ or, equivalently, $\frac{\nu(A)}{\mu_+(A)} = \frac{-\beta(A)}{\alpha(A)}$. So, in order to show that ν/μ_+ is a numerical representation of $\succsim$ on $\mathcal{P}_*(X)$, it suffices to show that $-\beta/\alpha$ represents $\succsim$ on $\mathcal{P}_*(X)$. Notice first that $-\beta/\alpha$ is well-defined because $-\beta(A)/\alpha(A) = \nu(A)/\mu_+(A)$ and $\mu_+(A) > 0$ for all $A \in \mathcal{P}_*(X)$. Pick any $A, B \in \mathcal{P}_*(X)$ with $A \succsim B$. By construction, $\nu^B(A) \geq 0$. From this, we derive $\alpha(B)\nu(A) + \beta(B)\mu_+(A) \geq 0$ and $\nu(A) \geq -\beta(B)\mu_+(A)/\alpha(B)$. We also have $\alpha(A)\nu(A) + \beta(A)\mu_+(A) = 0$ or, equivalently, $\nu(A) = -\beta(A)\mu_+(A)/\alpha(A)$. So, $-\beta(A)\mu_+(A)/\alpha(A) \geq -\beta(B)\mu_+(A)/\alpha(B)$ or, after simplification, $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. So, we have proved that $A \succsim B$ implies $-\beta(A)/\alpha(A) \geq -\beta(B)/\alpha(B)$. Proving the converse is done following the inverse path. This concludes the proof that ν/μ_+ is a numerical representation of $\succsim$ on $\mathcal{P}_*(X)$. $\square$

Theorem 1 *Assume that $\succsim$ is an ordering of $\mathcal{P}(X)$ that satisfy Richness. Then $\succsim$ satisfies Balancedness, Averaging and the Archimedean axiom. iff there are two mappings $u : X \to \mathbb{R}$ and $p : X \to \mathbb{R}^{++}$ such that* (1) *holds.*

Proof. If $\succsim$ is trivial, then (1) trivially holds with u constant. We therefore assume in the rest of the proof that $\succsim$ is not trivial.

Define $f : \mathcal{P}(X) \to \mathbb{R}$ by $f(A) = \frac{\nu(A)}{\mu_+(A)}$ for all $A \in \mathcal{P}(X)$. Remember that ν and μ_+ are disjoint-additive. Hence,

$$f(A) = \frac{\sum_{a\in A} \nu(\{a\})}{\sum_{a\in A} \mu_+(\{a\})} = \frac{\sum_{a\in A} f(\{a\})\mu_+(\{a\})}{\sum_{a\in A} \mu_+(\{a\})}.$$

Define two mappings $u : X \to \mathbb{R}$ and $p : X \to \mathbb{R}^{++}$ by $u(a) = f(\{a\})$ and $p(a) = \mu_+(\{a\})$ and we obtain

$$f(A) = \frac{\sum_{a\in A} u(a)p(a)}{\sum_{a\in A} p(a)}.$$

We already know (Lemma 14) that $A \succsim B$ iff $f(A) \geq f(B)$ for all $A, B \in \mathcal{P}_*(X)$. We now must prove that it also holds for $A, B \in \mathcal{P}(X)$. We consider several cases.

1. $A \in M(X)$ and $B \in \mathcal{P}_*(X)$. By Lemma 1, there is $B' \in \mathcal{P}(X)$ such that $B' \cap A = \varnothing$ and $B' \sim B$. By Lemma 14, $\nu(B)/\mu_+(B) = \nu(B')/\mu_+(B')$. By Averaging, $A \succ A \cup B' \succ B'$ and, hence, $A \cup B' \in \mathcal{P}_*(X)$. So,

   $$\frac{\nu(A \cup B')}{\mu_+(A \cup B')} = \frac{\nu(A) + \nu(B')}{\mu_+(A) + \mu_+(B')} > \frac{\nu(B')}{\mu_+(B')} = \frac{\nu(B)}{\mu_+(B)}.$$

   Since μ_+ is positive, this yields

   $$\frac{\nu(A)}{\mu_+(A)} > \frac{\nu(B)}{\mu_+(B)},$$

   in line with the fact that $A \succ B$.

2. $A \in m(X)$ and $B \in \mathcal{P}_*(X)$. Similar to the previous case.

3. $A, B \in m(X)$. Choose some $C \in \mathcal{P}_*(X)$ so that $C \cap (A \cup B) = \varnothing$. By Averaging, $B \cup C \succ B$ and, so, by transitivity, $B \cup C \succ A$. By case 2, $\nu(B \cup C)/\mu(B \cup C) > \nu(A)/\mu(A)$ and

   $$\frac{\nu(B) + \nu(C)}{\mu(B) + \mu(C)} > \frac{\nu(A)}{\mu(A)}. \tag{8}$$

   By Lemma 12, we can choose C in a given equivalence class of $\succsim$, with $\mu(C)$ arbitrarily close to zero. Since all sets in a given equivalence class have the same ratio ν/μ, we can actually choose C with $\mu(C)$ and $\nu(C)$ arbitrarily close to zero. Assume now for contradiction that $f(A) > f(B)$, that is, $\nu(A)/\mu(A) > \nu(B)/\mu(B)$. Then, if we choose C as described above, we clearly have

$$\frac{\nu(A)}{\mu(A)} > \frac{\nu(B)+\nu(C)}{\mu(B)+\mu(C)},$$

in contradiction of (8).

4. $A, B \in M(X)$. Similar to the previous case.

5. $A \in M(X)$ and $B \in m(X)$. Suppose $A \cap B = \varnothing$. Then $A \succ A \cup B \succ B$ and, hence, $A \cup B \in \mathcal{P}_*(X)$. From $A \succ A \cup B$ and case 1, we derive $f(A) > f(A \cup B)$. From $A \cup B \succ B$ and case 2, we derive $f(A \cup B) > f(B)$. By Transitivity, $f(A) > f(B)$ as required.

□

4 Some unresolved questions

4.1 Independence of the axioms

For the moment, given an environment we are not capable of showing the independence of the three axioms used in the characterization of the CEU family of criteria. However, we are capable of finding orderings of $\mathcal{P}(X)$ that do not belong to the CEU family but that satisfy averaging, balancedness and richness (but that violate the Archimedean axiom). Here is the example.

Example 1 *Let $X = \mathbb{R}^2_{++} \times \mathbb{R}^2$. For every $A \in \mathcal{P}(X)$, define*

$$U_1(A) = \frac{\sum_{a \in A} a_1 a_3}{\sum_{a \in A} a_1}$$

and

$$U_2(A) = \frac{\sum_{a \in A} a_2 a_3}{\sum_{a \in A} a_2}.$$

Define then $\succsim$ on $\mathcal{P}(X)$ by

$$A \sim B \iff U_1(A) = U_1(B) \text{ and } U_2(A) = U_2(B);$$

$$A \succ B \iff \begin{cases} U_1(A) > U_1(B) \\ or \\ U_1(A) = U_1(B) \text{ and } U_2(A) > U_2(B). \end{cases}$$

I first show that this ranking violates the Archimedean axiom. Let $A = \{(1,2,0,-1)\}$, $B = \{(1,1,0,-1)\}$, $A_i = \{(1,2,0,i)\}$, $B_i = \{(1,1,0,i)\}$, $C = \{(1,1,0,0)\}$ and

$D = \{(2,1,0,0)\}$. We clearly have $A \sim A_i \sim B \sim B_i \sim C \sim D$ for all $i \in \mathbb{N}$. Let $F = \{(0,0,-1,0)\}$. We have $A \succ F$, $A \cup F \succ B \cup F$, $A_i \cup F \sim A \cup F$ and $B_i \cup F \sim B \cup F$ for all $i \in \mathbb{N}$. Yet, $C \cup F \bigcup_{i=0}^{n} A_i \prec D \cup F \bigcup_{i=0}^{n} B_i$ for all $n \in \mathbb{N}$. We now show that $\succsim$ satisfies Averaging. Suppose first that $A \succ B$. Using the definition of $\succsim$, this is either equivalent to:

$$
\begin{gathered}
U_1(A) > U_1(B) \\
\Longleftrightarrow \\
U_1(A) > U_1(A \cup B) > U_1(B) \\
\Longleftrightarrow \\
A \succ A \cup B \succ B
\end{gathered}
$$

or to:

$$
\begin{aligned}
& U_1(A) = U_1(B) \text{ and } U_2(A) > U_2(B) \\
& \Longleftrightarrow \\
& U_1(A) = U_1(A \cup B) = U_1(B) \text{ and } U_2(A) > U_2(A \cup B) > U_2(B) \\
& \Longleftrightarrow \\
& A \succ A \cup B \succ B.
\end{aligned}
$$

A similar reasoning holds when $A \sim B$. To show that $\succsim$ satisfies Richness, consider $A, B, C \in \mathcal{P}(X)$ such that $A \succ B \succ C$. We will show that there exists a set $D = \{d, e\}$ such that $D \cap (A \cup C) = \emptyset$, $D \sim A$ and $D \cup C \sim B$. So, we must have

$$\frac{d_1 d_3 + e_1 e_3}{d_1 + e_1} = U_1(A), \tag{9}$$

$$\frac{d_2 d_3 + e_2 e_3}{d_2 + e_2} = U_2(A), \tag{10}$$

$$\frac{d_1 d_3 + e_1 e_3 + \sum_{c \in C} c_1 c_3}{d_1 + e_1 + \sum_{c \in C} c_1} = U_1(B), \tag{11}$$

$$\frac{d_2 d_3 + e_2 e_3 + \sum_{c \in C} c_2 c_3}{d_2 + e_2 + \sum_{c \in C} c_2} = U_2(B). \tag{12}$$

Set $d_3 = \max(U_1(A), U_2(A)) + 1$ and $e_3 = \min(U_1(A), U_2(A)) - 1$. There clearly exist $d_1, e_1 \in \mathbb{R}_{++}$ such that (9) holds. Notice that d_1, e_1 are not unique; they can be scaled by any positive constant and we can choose this constant so that (11) holds. Similarly, there clearly exist $d_2, e_2 \in \mathbb{R}_{++}$ such that (10) holds. They are unique up to a multiplication by a positive constant, that we can choose independently of the scaling constant for d_1, e_1. So, we can choose it so that (12) holds. In order to guarantee that $D \cap (A \cup C) = \emptyset$, we can freely manipulate d_4 and e_4. Hence Richness holds. Finally, to show that $\succsim$ satisfies Balancedness, *consider finite and non-empty subsets A, B, C, D of X such that $A \sim B \succ C, D$ and $(A \cup B) \cap (C \cup D) = \emptyset$. We have $A \cup C \succsim B \cup C$ if and only if either: $U_1(A \cup C) > U_1(B \cup C)$ iff $U_1(A \cup D) > U_1(B \cup D)$ iff $A \cup D \succsim B \cup D$ or $[U_1(A \cup C) = U_1(B \cup C)$ and $U_2(A \cup C) \geq U_2(B \cup C)]$ iff $[U_1(A \cup D) = U_1(B \cup D)$ and $U_2(A \cup D) \geq U_2(B \cup D)]$ iff $A \cup D \succsim B \cup D$.*

We are also capable, as shown in the next example, of finding non CEU orderings that satisfy balancedness, richness and the Archimedean axiom but that violate averaging.

Example 2 *Let $X = \mathbb{R}_{++} \times \mathbb{R}^2$, $p(x) = x_1$, $u(x) = x_2$,*

$$U(A) = \frac{\sum_{a \in A} p(a)u(a)}{\left(\sum_{a \in A} p(a)\right)^2}$$

and $A \succsim B$ iff $U(A) \geq U(B)$.
The ranking $\succsim$ clearly satisfies Richness and the Archimedean axiom. It violates Averaging because $A = \{(3/4, 2, 0)\} \sim B = \{(3/4, 2, 1)\} \succ A \cup B$.
Let us prove that $\succsim$ satisfies Balancedness. $A \sim B$ implies:

$$\sum_{a \in A} p(a)u(a) \left(\sum_{b \in B} p(b)\right)^2 = \left(\sum_{a \in A} p(a)\right)^2 \sum_{b \in B} p(b)u(b). \tag{13}$$

while $A \cup C \succsim B \cup C$ implies:

$$\left(\sum_{a \in A} p(a)u(a) + \sum_{c \in C} p(c)u(c)\right) \left(\left(\sum_{b \in B} p(b)\right)^2 + \left(\sum_{c \in C} p(c)\right)^2\right)$$
$$\geq \left(\sum_{b \in B} p(b)u(b) + \sum_{c \in C} p(c)u(c)\right) \left(\left(\sum_{a \in A} p(a)\right)^2 + \left(\sum_{c \in C} p(c)\right)^2\right)$$

or, after distributing:

$$\sum_{a \in A} p(a)u(a) \left(\sum_{b \in B} p(b)\right)^2 + \sum_{c \in C} p(c)u(c) \left(\sum_{b \in B} p(b)\right)^2 + \sum_{a \in A} p(a)u(a) \left(\sum_{c \in C} p(c)\right)^2$$
$$\geq \sum_{b \in B} p(b)u(b) \left(\sum_{a \in A} p(a)\right)^2 + \sum_{c \in C} p(c)u(c) \left(\sum_{a \in A} p(a)\right)^2 + \sum_{b \in B} p(b)u(b) \left(\sum_{c \in C} p(c)\right)^2. \tag{14}$$

Substituting (13) *in* (14) *yields:*

$$\sum_{c \in C} p(c)u(c) \left(\sum_{b \in B} p(b)\right)^2 + \sum_{a \in A} p(a)u(a) \left(\sum_{c \in C} p(c)\right)^2$$
$$\geq \sum_{c \in C} p(c)u(c) \left(\sum_{a \in A} p(a)\right)^2 + \sum_{b \in B} p(b)u(b) \left(\sum_{c \in C} p(c)\right)^2$$

or:

$$\sum_{c\in C} p(c)u(c)\left(\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2\right) \geq \left(\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)\right)\left(\sum_{c\in C} p(c)\right)^2.$$

Since $\left(\sum_{c\in C} p(c)\right)^2 > 0$*, one obtains:*

$$\frac{\sum_{c\in C} p(c)u(c)}{\left(\sum_{c\in C} p(c)\right)^2} \geq \frac{\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)}{\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2} = \frac{\sum_{b\in B} p(b)u(b)}{\left(\sum_{b\in B} p(b)\right)^2} \tag{15}$$

if $\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2 > 0$ *or*

$$\frac{\sum_{c\in C} p(c)u(c)}{\left(\sum_{c\in C} p(c)\right)^2} \leq \frac{\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)}{\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2} = \frac{\sum_{b\in B} p(b)u(b)}{\left(\sum_{b\in B} p(b)\right)^2} \tag{16}$$

if $\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2 < 0$*. Inequality* (15) *is not possible because* $B \succ C$*. We therefore conclude that* (16) *holds and* $\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2 \leq 0$*. We also know that* $D \prec B$*. This implies*

$$\frac{\sum_{d\in D} p(d)u(d)}{\left(\sum_{d\in D} p(d)\right)^2} \leq \frac{\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)}{\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2} = \frac{\sum_{b\in B} p(b)u(b)}{\left(\sum_{b\in B} p(b)\right)^2}.$$

Hence:

$$\sum_{d\in D} p(d)u(d)\left(\left(\sum_{b\in B} p(b)\right)^2 - \left(\sum_{a\in A} p(a)\right)^2\right) \geq \left(\sum_{b\in B} p(b)u(b) - \sum_{a\in A} p(a)u(a)\right)\left(\sum_{d\in D} p(d)\right)^2.$$

and

$$\begin{aligned}\sum_{d\in D} p(d)u(d)\left(\sum_{b\in B}(b)\right)^2 &+ \sum_{a\in A} p(a)u(a)\left(\sum_{d\in D} p(d)\right)^2 \\ &\geq \sum_{d\in D} p(d)u(d)\left(\sum_{a\in A} p(a)\right)^2 + \sum_{b\in B} p(b)u(b)\left(\sum_{d\in D} p(d)\right)^2.\end{aligned}$$

If we add (13) *to this inequality, we obtain*

$$\begin{aligned}&\sum_{a\in A} p(a)u(a)\left(\sum_{b\in B} p(b)\right)^2 + \sum_{d\in D} p(d)u(d)\left(\sum_{b\in B} p(b)\right)^2 + \sum_{a\in A} p(a)u(a)\left(\sum_{d\in D} p(d)\right)^2 \\ &\geq \sum_{b\in B} p(b)u(b)\left(\sum_{a\in A} p(a)\right)^2 + \sum_{d\in D} p(d)u(d)\left(\sum_{a\in A} p(a)\right)^2 + \sum_{b\in B} p(b)u(b)\left(\sum_{d\in D} p(d)\right)^2.\end{aligned}$$

Let us now add $\sum_{d\in D} p(d)u(d) \left(\sum_{d\in D} p(d)\right)^2$ *on both sides and factorize. We obtain*

$$\left(\sum_{a\in A} p(a)u(a) + \sum_{d\in D} p(d)u(d)\right) \left(\left(\sum_{b\in B} p(b)\right)^2 + \left(\sum_{d\in D} p(d)\right)^2\right)$$
$$\geq \left(\sum_{b\in B} p(b)u(b) + \sum_{d\in D} p(d)u(d)\right) \left(\left(\sum_{a\in A} p(a)\right)^2 + \left(\sum_{d\in D} p(d)\right)^2\right)$$

which implies $A \cup D \succsim B \cup D$*. This concludes the proof that* $\succsim$ *satisfies Balancedness.*

However, we were not able to find examples of orderings of $\mathcal{P}(X)$ that satisfy the Archimedean axiom and Averaging but that violate Balancedness.

4.2 Some unpleasant implications of our richness condition

The richness condition that we use to provide our characterization is strong, and seems to impose some additional condition on the functions p and u that are used in the representation of any CEU criterion. Yet, we are not for the moment capable to analytically identify what these conditions can be. We can not either provide a topological interpretation of our characterization result in the same spirit than the one we obtain in [2]. An example of the implication of our richness condition is provided in the following proposition, where we show that if $X = \mathbb{R}$ (for instance consequences of a decision under ignorance are amounts of money), then it is impossible with our richness condition to have both the functions p and the function u to be monotonic if the function u is continuous.

Proposition 1 *Suppose that* $X = \mathbb{R}$*. Then if* $\succsim$ *is a CEU ranking satisfying richness, then, if the function* u *in expression (1) is continuous, it can not be monotonic if* p *is monotonic.*

Proof: Suppose A, B, C are three finite and non-empty subsets of X such that $A \succ B \succ C$ or $A \prec B \prec C$. Richness implies the existence of a set D disjoint from A and C such that $A \sim D$ and $D \cup C \sim B$. For any set $E \in \mathcal{P}(X)$, define $U(E)$ by

$$U(E) = \frac{\sum_{e\in E} p(e)u(e)}{\sum_{e\in E} p(e)}.$$

Then

$$U(D) = \frac{\sum_{d\in D} p(d)u(d)}{\sum_{d\in D} p(d)} = U(A) \tag{17}$$

and

$$U(D \cup C) = \frac{\sum_{d \in D} p(d)u(d) + \sum_{c \in C} p(c)u(c)}{\sum_{d \in D} p(d) + \sum_{c \in C} p(c)} = U(B).$$

This last equation can be rewritten as

$$\sum_{d \in D} p(d)u(d) + \sum_{c \in C} p(c)u(c) = U(B) \left(\sum_{d \in D} p(d) + \sum_{c \in C} p(c) \right). \tag{18}$$

From (17), we obtain $\sum_{d \in D} p(d)u(d) = U(A) \sum_{d \in D} p(d)$. By definition of U, we also have $\sum_{c \in C} p(c)u(c) = U(C) \sum_{c \in C} p(c)$. If we replace in (18), we find

$$U(A) \sum_{d \in D} p(d) + U(C) \sum_{c \in C} p(c) = U(B) \left(\sum_{d \in D} p(d) + \sum_{c \in C} p(c) \right).$$

Put differently,

$$(U(A) - U(B)) \left(\sum_{d \in D} p(d) \right) = (U(B) - U(C)) \left(\sum_{c \in C} p(c) \right)$$

or

$$\frac{U(A) - U(B)}{U(B) - U(C)} = \frac{\sum_{c \in C} p(c)}{\sum_{d \in D} p(d)}. \tag{19}$$

Remember that this holds for any A, B, C. In particular, for any $B = \{b\}$. Thanks to the continuity of u, we can choose b so that $U(B) = u(b)$ is between $U(C)$ and $U(A)$ and is as close as we want to $U(A)$ or $U(C)$. We can therefore make the ratio in the left-hand side of (19) as close to 0 or ∞ as we wish. So, with A and C given, Richness implies the existence of a set D with $\sum_{d \in D} p(d)$ arbitrary close to 0 or ∞.

Suppose p is non-decreasing. If we want to make $\sum_{d \in D} p(d)$ arbitrary close to 0, then $\max_{d \in D} p(d)$ must be arbitrary close to 0. This implies that $\lim_{x \to \inf X} p(x) = 0$ and, hence, $\max_{d \in D} d$ must be arbitrary close to $\inf X$.

- If u is non-decreasing, then $U(D) < U(A)$ (if we have chosen $A \succ C$). This contradicts (17) and proves that u continuous and non-decreasing is not compatible with p non-decreasing.
- If u is non-increasing, then $U(D) > U(A)$ (if we have chosen $A \prec C$). This contradicts (17) and proves that u continuous and non-increasing is not compatible with p non-decreasing.

Suppose p is non-increasing. If we want to make $\sum_{d \in D} p(d)$ arbitrary close to ∞, then $\min_{d \in D} p(d)$ must be arbitrary large. This implies that $\lim_{x \to \sup X} p(x) = \infty$ and, hence, $\min_{d \in D} d$ must be arbitrary close to $\sup X$.

- If u is non-decreasing, then $U(D) > U(A)$ (if we have chosen $A \prec C$). This contradicts (17) and proves that u continuous and non-decreasing is not compatible with p non-increasing.

- If u is non-increasing, then $U(D) < U(A)$ (if we have chosen $A \succ C$). This contradicts (17) and proves that u continuous and non-increasing is not compatible with p non-increasing. QED

In the next proposition, we establish that if X is a topological space (for instance a separable one of the kind considered in [2]), then no Uniform Expected Utility criterion in which u is a continuous utility function satisfies the richness condition. This shows that the characterization of the CEU family of criteria that we provide in this paper does not contain all members of that family because it excludes, at least in topological environment, the UEU subclass of that family that is obtained by considering only constant functions p and continuous functions u.

Proposition 2 *Let X be a topological space, and let $\succsim$ is a non-trivial UEU ranking with u continuous. Then $\succsim$ violates the Richness condition.*

Proof: Since $\succsim$ is not trivial, there are $A = \{a\}, C = \{c\}$ with $a, c \in X$ such that $u(a) > u(c)$. Let D be a set such that $D \sim A$. The set D can be a singleton ($D = \{d\}$ with $u(d) = u(a)$) or a set with several elements. If D is a singleton, then $(D \cup C) = (u(a) + u(c))/2$. If D is not a singleton, then $(D \cup C) > (u(a) + u(c))/2$. So, for all $D \sim A$, $(D \cup C) \geq (u(a) + u(c))/2$. The continuity of u implies that, for any real number α between $u(a)$ and $u(c)$, there exists $B = \{b\} \in \mathcal{P}(X)$ such that $u(b) = \alpha$. If we choose α strictly smaller than $(u(a) + u(c))/2$, then $(D \cup C) > (B)$ and $D \cup C \succ B$, for any D with $D \sim A$. Hence, Richness does not hold. QED.

4.3 Uniqueness of the functions u and p

To be provided

5 Comparative Ignorance or Ambiguity aversion

To be provided

6 Conclusion

To be provided

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