Pseudo-Riemannian submersions from complex pseudo-hyperbolic space forms *

Paul-Andi Nagy

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Abstract

We study pseudo-Riemannian submersions with complex fibers from a complex pseudo-hyperbolic space form. Using the effect of the Kähler condition on the fundamental O'Neill tensors we prove total geodesicity for all such submersions. Simple geometric arguments are given in order to obtain a classification.

Contents

1 Introduction

Contrary to Riemannian submersions, for which a considerable amount of information is now available, pseudo-Riemannian submersions, as defined in [13] were subject to less attention. However, various classification results are known, such as the description of totally geodesic pseudo-Riemannian submersions from anti-de Sitter space [11] or from pseudo-hyperbolic spaces into a Riemannian manifold [2].

In this paper we study pseudo-Riemannian submersions with complex fibers from a complex pseudo-hyperbolic space form. We are motivated by results in [12], asserting that on a compact Kähler manifold any Riemannian foliation with complex leaves is totally geodesic or transversally integrable, provided that the manifold is holonomy irreducible. It is then natural to investigate the same problem in the simplest non-compact case, that of constant negative holomorphic curvature. Our first main result is as follows.

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Theorem 1.1 *Let $\pi : M \to N$ be a pseudo-Riemannian submersion with complex fibers from an indefinite Kähler manifold of constant negative holomorphic curvature. Then the following hold :*
(i) The fibers of π are totally geodesic;
(ii) If the fibers of π are totally geodesic then they are 2-dimensional;
(iii) π is equivalent to the twistor fibration of a quaternionic Kähler manifold of constant negative quaternionic curvature. Moreover, when M is simply connected and complete one gets the canonical Hopf submersion $\mathbb{C}H^{2n+1}_{2s+1} \to \mathbb{H}H^{2n}_{2s}$.

Hence, the complexity of the leaves together with the presence of a pseudo-Kähler metric is sufficient to ensure total geodesicity in this case. Howewer, the (very simple) proof of this fact is strongly using the constancy of the holomorphic sectional curvature, hence do not apply to other examples. Note also that in the case of the complex pseudo-hyperbolic space and under the assumption of total geodesicity, part (iii) of theorem was already proven in [2] if the base manifold is Riemannian and in [1] under the assumptions that the fibers are 2-dimensional or the base manifold is isotropic.

Theorem 1.1 has the following consequence related to (indefinite) Kähler submersions (see [16] for basic definitions).

Corollary 1.1 *They are no Kähler submersions starting from a (Riemannian) complex space form of non-zero scalar curvature. The same holds in the indefinite case.*

For the proof of this fact it suffices to assume, by eventually reversing the metric, that the scalar curvature is negative. Recall that such a submersion has integrable horizontal distribution (the proof of this, given in [16]) works also in the indefinite case). Using theorem 1.1, we obtain that the fibers are totally geodesic hence the total space of the submersion is locally a (pseudo)-Riemannian product, and we conclude by the indecomposability of a complex space form of non-zero scalar curvature. Note that in the Riemannian case and for positive scalar curvature corollary 1.1 was already proven in [7].

The paper is organised as follows. In section 2 we prove some usefull relations between the O'Neill's tensors of a pseudo-Riemannian submersion from a pseudo-Kähler manifold. Total geodesicity is studied in section 3, with special emphasis on the case of fibers of codimension 2, which appears to stand out in the discussion. In the last section we are concerned with showing that the fibers have to be two-dimensional. This a consequence of the single fact that the fibers have constant sectional holomorphic curvature and not of the constancy of the ambient holomorphic curvature.

2 Preliminaries

Let us first recall the definition of a pseudo-Riemannian submersion.

Definition 2.1 *[13] Let (M, g) and (N, h) be pseudo-Riemannian manifolds. A pseudo-Riemannian submersions $\pi : M \to N$ is a submersion such that the fibers are pseudo-Riemannian manifolds and such that $d\pi$ preserves the scalar product of vector normal to fibers.*

Given such an object let $\mathcal{V}$ be the vertical distribution. A direct consequence of the definition is that we have a direct sum $TM = \mathcal{V} \oplus H$ where H is the orthogonal complement of $\mathcal{V}$ in TM.

We start by collecting a number of basic facts about pseudo-Riemannian submersions and next we will specialize to the Kähler case. Let ∇ be the Levi-Civita connection of the metric g. Throughout this paper we will denote by $U, V, W.W'$ vector fields in $\mathcal{V}$ and by X, Y, Z etc. vector fields in H. It is easy to verify that the formula [14]

$$\overline{\nabla}_E F = (\nabla_E F_{\mathcal{V}})_{\mathcal{V}} + (\nabla_E F_H)_H$$

defines a metric connection with torsion on M (here the subscript denotes orthogonal projection on the subspace). The main property of this connection is that it preserves the distributions $\mathcal{V}$ and H. If T and A are the O'Neill's tensors of the submersion π then the following relations between ∇ and $\overline{\nabla}$ are known to hold

$$\begin{aligned} \nabla_X Y = \overline{\nabla}_X Y + A_X Y,\ \nabla_X V = \overline{\nabla}_X V + A_X V \\ \nabla_V X = \overline{\nabla}_V X + T_V X,\ \nabla_V W = \overline{\nabla}_V W + T_V W. \end{aligned}$$

For the algebraic properties of $T$and A see [4]. We only recall here that A is skew-symmetric on H while T is symmetric on $\mathcal{V}$.

Our workhorse device in what follows will be the O'Neill's formula for a pseudo-Riemannian submersion which we recall below. A proof for the Riemannian case extending over verbatim to the pseudo-Riemannian one can be find in [4].

Proposition 2.1
(i) $R(U, V, W, W') = \overline{R}(U, V, W, W') - < T_U W, T_V W' > + < T_V W, T_U W' >$
(ii) $R(U, V, W, X) = < (\overline{\nabla}_V T)(U, W), X > - < (\overline{\nabla}_U T)(V, W), X >$
(iii) $R(X, U, Y, V) = < (\overline{\nabla}_X T)(U, V), Y > + < (\overline{\nabla}_U A)(X, Y), V > -$
$< T_U X, T_V Y > + < A_X U, A_Y V >$
(iv) $R(V, W, X, Y) = < (\overline{\nabla}_V A)(X, Y), W > - < (\overline{\nabla}_W A)(X, Y), V > +$
$< A_X V, A_Y W > - < A_X W, A_Y V > - < T_V X, T_W Y > + < T_W X, T_V Y >$
(v) $R(X, Y, Z, V) = < (\overline{\nabla}_Z A)(X, Y), V > + < A_X Y, T_V Z > - < A_Y Z, T_V X > -$
$< A_Z X, T_V Y >$
(vi) $R(X, Y, Z, Z') = \overline{R}(X, Y, Z, Z') - 2 < A_X Y, A_Z Z' > + < A_Y Z, A_X Z' > -$
$< A_X Z, A_Y Z' >$.

Here R and $\overline{\nabla}$ are the curvature tensors of the connexions ∇ and $\overline{\nabla}$, respectively.

In the rest of this section we will asssume that (M, g) is an indefinite Kähler manifold of dimension $2m$, with complex structure J(see [3] for some basic examples). Moreover, we suppose that the submersion π has complex fibers, that is $J\mathcal{V} = \mathcal{V}$ (and then, of course, $JH = H$). As $\nabla J = 0$, it follows that $\overline{\nabla} J = 0$, hence we obtain information about the complex type of the tensors A and T as follows

$$\begin{aligned} A_X(JY) = J(A_X Y), \quad A_{JX} V = -J(A_X V) = A_X(JV) \\ T_{JV} W = J(T_V W), \quad T_{JV} X = -J(T_V X) = T_V(JX). \end{aligned} \tag{2.1}$$

We also have $A_{JX}JY = -A_XY$ and $T_{JV}JW = -T_VW$. An immediate consequence of these relations is that the fibers of our submersion are minimal, that is the mean curvature vectore field vanishes.

We will use now the Kähler structure on M, together with suitable curvature identities to get some geometric information about the tensors A and T.

Lemma 2.1 *Let X, Y, Z be in H and V, W in $\mathcal{V}$. Then we have :*
(i) $(\overline{\nabla}_X A)(Y, Z) = 0$
(ii) $< A_XY, T_VZ >= 0$
(iii) $< (\overline{\nabla}_V A)(X, Y), W >=< (\overline{\nabla}_W A)(X, Y), V >$.
(iv) $2 < (\overline{\nabla}_V A)(X, Y), W >=< (\overline{\nabla}_Y T(V, W), X > - < (\overline{\nabla}_X T(V, W), Y >$.

Proof :
We will prove (i) and (ii) simultaneously. Since (M, g) is Kähler one has $R(JX, JY, Z, V) = R(X, Y, Z, V)$. Hence by (2.1) and (v) of proposition 2.1 we easily arrive at $< (\overline{\nabla}_X A)(Y, Z), V > + < A_XY, T_VZ >= 0$. But we know that (see [4], page 242)

$$\sigma_{X,Y,Z} < (\overline{\nabla}_X A)(Y, Z), V >= \sigma_{X,Y,Z} < A_XY, T_VZ >$$

(here σ denotes the cyclic sum) thus $\sigma_{X,Y,Z} < A_XY, T_VZ >= 0$ and further

$$R(X, Y, Z, V) =< A_XY, T_VZ > .$$

Using again that $R(JX, JY, T, V) = R(X, Y, Z, V)$ and (2.1) yields (ii), hence (i) follows. To prove (iii) we use the fact that $R(V, W, JX, JY) = R(V, W, X, Y)$, (iv) of proposition 2.1 and (2.1). The proof of (iv) follows by the formula

$$< (\overline{\nabla}_W A)(X, Y), V > + < (\overline{\nabla}_V A)(X, Y), W >=< (\overline{\nabla}_Y T(V, W), X > - < (\overline{\nabla}_X T(V, W), Y >$$

(see [4], page 242) and (iii) ■

Remark 2.1 *(i) By the the first two assertions of lemma 2.1 we obtain that $R(X, Y, Z, V) = 0$, a condition frequentely imposed when studying Riemannian foliations (see chapter 5 of [14] and references therein).*
(ii) By (i) and (ii) of the previous lemma it is easy to see that H satisfies the Yang-Mills condition.

Another result that will be needed in the next section is the following :

Lemma 2.2 *We have :*
(i)

$$\overline{R}(X, Y)V = 2[A_X, A_Y]V + Q(X, Y)V$$

for all X, Y in H and V in $\mathcal{V}$ where we defined $Q(X, Y)V = T_{T_VY}X - T_{T_VX}Y$.
(ii) If the submersion has totally geodesic fibers, then $\overline{R}(X, V_1, V_2, V_3) = 0$.

The proof follows from the general formulas in [14], page 100, and lemma 2.1, (iii).

3 Total geodesicity

We will suppose now that our manifold M has constant holomorphic curvature $c < 0$. An example is $\mathbb{C}H^n_s$, the complex pseudo-hyperbolic space of real dimension $2n$, endowed with its canonical metric of signature $2s$ and of constant holomorphic sectional curvature, equal to $c < 0$ (see [3]), or any of its smooth quotients. The aim of this section is to prove that every pseudo-Riemannian submersion from M has totally geodesic fibers.

Recall now that the curvature tensor R of M equals cR_0 where

$$4R_0(E_1, E_2, E_3, E_4) = < E_1, E_3 >< E_2, E_4 > - < E_1, E_4 >< E_2, E_3 > + \\ < E_1, JE_3 >< E_2, JE_4 > - < E_1, JE_4 >< E_2, JE_3 > +2 < E_1, JE_2 >< E_3, JE_4 > .$$

Using this and the lemma 2.1, (ii) one can give a very simple proof of the fact that the fibers of π are totally geodesic in codimension at least 4, as follows.

Proposition 3.1 *Every pseudo-Riemannian submersion with complex fibers from the complex pseudo-hyperbolic space has totally geodesic fibers, provided that the real dimension of N is at least 4.*

Proof :
Using (iii) of proposition 2.1 and (iv) of lemma 2.1 we obtain

$$cR_0(X, U, Y, V) = \frac{1}{2}(< (\overline{\nabla}_X T)(U, V), Y > + < (\overline{\nabla}_Y T)(U, V), X >) + \\ < A_X U, A_Y V > - < T_U X, T_V Y >$$

Or $R_0(X, JU, Y, JV) = R_0(X, U, Y, V)$ hence by (2.1) we get that :

3.1
$$< A_X U, A_Y V > - < T_U X, T_V Y >= \\ k(< X, Y >< U, V > + < X, JY >< U, JV >)$$

where $k = \frac{c}{4}$. Taking $U = T_W Z$ we obtain by lemma 2.1, (ii) and after some simple manipulations :

$$T(T_W Z, T_V Y) = k(< T_W Z, V > Y + < T_W Z, JV > JY).$$

Taking $V = W$ and permuting the roles of Z and Y we obtain by the symmetry of T and since $k \neq 0$:

$$< T_V Z, V > Y + < T_V Z, JV > JY =< T_V Y, V > Z + < T_V Y, JV > JZ$$

or further

$$< Z, v_0 > Y + < Z, Jv_0 > JY =< Y, v_0 > Z + < Y, Jv_0 > JZ$$

where $v_0 = T_V V$. Using the fact that the dimension of H is at least 4 it is now an elementary exercise to conclude the vanishing of v_0, hence that of T ■

The rest of this section will be concerned with showing that the case when the (real) dimension of the horizontal distribution equals 2 cannot occur.

Let us define, for each V in $\mathcal{V}$ an operator $P_V : H \to H$ by setting $P_V X = A_X V$ whenever X is in H. Then P_V is skew-symmetric and furthermore, $P_V J + JP_V = 0$ for each V belonging to $\mathcal{V}$.

From now on we will assume that H is two-dimensional. Then the above algebraic properties of the operator P_V, V in $\mathcal{V}$ imply its vanishing hence $A = 0$. Hence both distributions $\mathcal{V}$ and $\mathcal{H}$ are integrable.

Remark 3.1 *It is easy to see that under the above conditions J projects onto a Kähler structure on N giving π the structure of a Kähler submersion in the sense of [16] (see also [?]).*

Since A vanishes (3.1) becomes

3.2 $$< T_U X, T_V Y >= -k(< X, Y >< U, V > + < X, JY >< U, JV >).$$

We will now examine our geometric data locally. For simplicity, let us assume that $c = -4$, so that $k = -1$. As H is two-dimensional either g is positive or it is negative on H. Let e be a local horizontal vector field with $g(e, e) = \varepsilon$ where $\varepsilon \in \{\pm 1\}$. Then $\{e, Je\}$ gives a local basis of H and expressing T in this basis we obtain $T = \alpha \cdot e + \beta \cdot Je$ where α and β are symmetric forms on $\mathcal{V}$. Let F be the symmetric (with respect to g) endomorphism of $\mathcal{V}$ associated to α. As by definition $T_V e = -\varepsilon FV$ whenever V is in $\mathcal{V}$, (3.2) gives $F^2 = \varepsilon \cdot 1_{\mathcal{V}}$.

Lemma 3.1 *(i) We have $(\overline{\nabla}_U F)V = 0$ for all U, V in $\mathcal{V}$.*
(ii) The fibers of π are flat in the induced metric.

Proof :
(i) The constant holomorphic curvature assumption implies that $R(U, V, W, X) = 0$ and this yields further by using (ii) of proposition 2.1 that $(\overline{\nabla}_V T)(U, W) = (\overline{\nabla}_U T)(V, W)$. We now choose e to be basic and since this ensures the vanishing of $\overline{\nabla}_V e, \overline{\nabla}_V Je$ we arrive at $(\overline{\nabla}_V F)U = (\overline{\nabla}_U F)V$. But, for all V in $\mathcal{V}$ the symmetry of F implies that $\overline{\nabla}_V F$ is a symmetric tensor. Moreover, $\overline{\nabla}_V F$ anticommutes with F (as F^2 is a constant multiple of the identity) hence $(\overline{\nabla}_V F)F$ is skew-symmetric. Or, the symmetry of $\overline{\nabla} F$ implies that the symmetric endomorphism $\overline{\nabla}_{FV} F$ equals $(\overline{\nabla}_V F)F$ and we conclude that $\overline{\nabla}_V F = 0$ for all V belonging to $\mathcal{V}$, in other words, F is $\overline{\nabla}$-parallel in $\mathcal{V}$.
(ii) Using (i) of proposition 2.1 we obtain after an easy calculation :

3.3 $$\begin{aligned}&\overline{R}(U, V, W, W') = \\ &-4(R_0(U, V, W, W') - \varepsilon R_0(U, V, FW, FW')) + 2g(U, JV)g(W, JW').\end{aligned}$$

for all U, V, W, W' in $\mathcal{V}$. But, since F is symmetric with $F^2 = \varepsilon$, it follows that $\overline{R}(U, V, FW, FW') = -\varepsilon \overline{R}(U, V, W, W')$. On the other hand, F is parallel in $\mathcal{V}$, hence it commutes with operators of the form $\overline{R}(U, V)$ and this yields to $\overline{R}(U, V, FW, FW') = \varepsilon \overline{R}(U, V, W, W')$. This proves the vanishing of the curvature of the fibers ■

We are now able to give a very simple, completely algebraic proof of the fact that the fibers cannot be of codimension 2.

Proposition 3.2 *There are no pseudo-Riemannian submersions with complex leaves and codimension 2 fibers from an indefinite complex space form of negative scalar curvature.*

Proof :
Using (ii) of the previous lemma we see that (3.3) becomes

3.4 $$R_0(U, V, W, W') - \varepsilon R_0(U, V, FW, FW') = \tfrac{1}{2} g(U, JV)g(W, JW').$$

for all U, V, W, W' in $\mathcal{V}$. We set $V = JU, W = U$ and we give W' succesively the values $JU, FU, (FJ)U$. After a short computation, we obtain the equations $g(U, FU)^2 + g(U, (FJ)U)^2 = -2\varepsilon g(U, U)^2$, $g(U, U)g(U, (FJ)U) = 0$ and $g(U, U)g(U, FU) = 0$.

These obviously imply that $g(U,U)=0$ in other words $\mathcal{V}$ is totally isotropic, a contradiction ■

Part (i) of the theorem 1.1 is now proven. Note that no completeness assumption or topological hypothesis on the manifold M is needed for proving the results in this section.

4 The geometry of the fiber

In the geometrical context described in the previous section let us suppose the fibers of the submersion be totally geodesic, i.e. $T=0$. Using (i) and (iv) of lemma 2.1 we obtain that A must be $\overline{\nabla}$-parallel. This easily implies

4.1 $$\overline{\nabla}_E(P_V X) = P_V(\overline{\nabla}_E X) + P_{\overline{\nabla}_E V} X$$

for all E in TM, where P is the operator defined in the proof of lemma 3.1. The following result shows how the curvature of the fibers can be computed using the tensor P.

Proposition 4.1

$$P_{R(V_1,V_2)V_3} = [P_{V_3}, [P_{V_1}, P_{V_2}]].$$

Proof :
Using the second Bianchi identity for the Hermitian connection $\overline{\nabla}$ (see [9]) we obtain

$$(\overline{\nabla}_X\overline{R})(Y,V_1,V_2,V_3) + (\overline{\nabla}_Y\overline{R})(V_1,X,V_2,V_3) + (\overline{\nabla}_{V_1}\overline{R})(X,Y,V_2,V_3) +$$
$$\overline{R}((\nabla_X J)JY,V_1,V_2,V_3) + \overline{R}((\nabla_Y J)JV_1,X,V_2,V_3) + \overline{R}((\nabla_{V_1} J)JX,Y,V_2,V_3) = 0.$$

Since the distributions $\mathcal{V}$ and H are $\overline{\nabla}$-parallel we get by lemma 2.2, (ii) that $(\overline{\nabla}_X\overline{R})(Y,V_1,V_2,V_3) = (\overline{\nabla}_Y\overline{R})(V_1,X,V_2,V_3) = 0$ and by (i) of the same lemma and (4.1) it can be seen that the term $(\overline{\nabla}_{V_1}\overline{R})(X,Y,V_2,V_3)$ also vanishes. Now, we compute the last two curvature terms in the Bianchi identity using lemma 2.2, (i) and the result follows by calculus ■

We will show now that if the fibers have constant holomorphic curvature then they are necessarily two-dimensional. In order to abbreviate notations we will formulate the main ingredient of the proof under the following form.

Lemma 4.1 *Let $(A,+,\cdot)$ be an associative algebra over $\mathbb{R}$ and suppose that x,y in A are subject to the following relations :*
(i) $x^3=\gamma kx, y^3=\varepsilon ky$
(ii) $xy^2+y^2x=\varepsilon kx$
(iii) $yx^2+x^2y=\gamma ky$
where k is a non-zero real number, and ϵ,γ are in $\{-1,1\}$. Then $x=y=0$.

Proof :
Multiplying (ii) at left and right by y we get $yxy^3+y^3xy=\varepsilon kyxy$ hence $yxy=0$ by (i). In the same way, but using this time (iii) one obtains $xyx=0$. Squaring now (ii) we get

$$(xy^2)^2+(y^2x)^2+xy^4x+y^2x^2y^2=k^2x^2.$$

It is easy to see that the first three terms are vanishing hence

$$\mathbf{4.2} \qquad y^2x^2y^2 = k^2x^2.$$

We now multiply (ii) at left by y^2x hence $y^2x^2y^2 + (y^2x)^2 = \varepsilon k y^2x^2$ and further $y^2x^2y^2 = \varepsilon k y^2 x^2$ since $(y^2x)^2 = 0$. Using (4.1) we get $\varepsilon y^2x^2 = kx^2$ thus right multiplication by x followed by (i) yields $\varepsilon y^2 x = kx$. Squaring, and using $yxy = 0$ we get $x^2 = 0$ and the conclusion now follows immediately ■

We are now able to prove the following.

Proposition 4.2 *Let $\pi : M \to N$ a pseudo-Riemannian submersion with totally geodesic fibers. Then the fibers are two-dimensional.*

Proof :
The fibers being totally geodesic they are of constant holomorphic sectional curvature equal to c. Then $R(V, JV)V = cg(V,V)JV$ for all V in $\mathcal{V}$. Using now proposition 4.1, we obtain

$$P_V^3 = -\frac{c}{4} g(V,V)P_V.$$

If U is in $\mathcal{V}$ such that $g(V,U) = g(V,JU) = 0$ then $R(V,JV)U = \frac{c}{2} g(V,V)JU$ and using proposition 4.1 we obtain

$$P_U P_V^2 + P_V^2 P_U = -\frac{c}{4} g(V,V) P_U.$$

Of course, permuting U and V it also follows that

$$P_V P_U^2 + P_U^2 P_V = -\frac{c}{4} g(U,U) P_V.$$

Now, if the dimension of the fibers is greater than 2, for each v in $\mathcal{V}$ with $g(v,v) = \varepsilon = \pm 1$ there exists u in $\mathcal{V}$ such that $g(v,u) = g(v,Ju) = 0$ and $g(u,u) = \gamma = \pm 1$. Apllying the previous lemma with $x = P_v, y = P_u$ we find that $P_v = 0$, in other words, the tensor A vanishes identically. Using (3.1) this implies easily that H is totally isotropic for g, an absurdity ■

Remark 4.1 *The proof of the previous result extends ad literam to show that the fibers of a pseudo-Riemannian submersion from a pseudo-Kähler manifold are two dimensional, provided they are complex, totally geodesic, and of constant holomorphic curvature $c \neq 0$.*

To complete the proof of theorem 1.1 we will need now the following

Lemma 4.2 *The restriction of g to $\mathcal{V}$ is negative definite.*

Proof :
Let us suppose that g is positive definite and choose a (locally) defined vector field of $\mathcal{V}$ with $g(e,e) = 1$. Consider the skew-symmetric endomorphism of H defined by $F = P_e$ and note that by (3.1) we have $F^2 = -k \cdot 1_H$. Hence H splits as the direct sum $H^+ \oplus H^-$ where $H^{\pm}$ are the eigenspaces of F coresponding to the eigenvalues $\pm\sqrt{-k}$. Moreover, $H^{\pm}$ are totally isotropic (F is skew-symmetric) and $H^- = JH^+$ (since $FJ + JF = 0$). Now, using (4.1) it is easy to show that

$$\overline{\nabla}_E X - \frac{1}{2}\alpha(E) JX$$

belongs to H^+ for all X, E in H^+ and TM respectively, where α is a local 1-form on M defined by $\alpha(E) = g(\overline{\nabla}_E e, Je), E$ in TM. Using this fact and the isotropy of H^+, an elementary computation shows that $2\overline{R}(X,Y,X,Y) = d\alpha(X,Y)g(X,JY)$ and since $d\alpha(X,Y) = -\overline{R}(X,Y,e,Je)$ we arrive throught lemma 2.2 at $\overline{R}(X,Y,X,Y) = 2kg(X,JY)^2$. M being of constant holomorphic curvature we have $R(X,Y,X,Y) = 3kg(X,JY)^2$ and a simple computation shows that $g(A_XY, A_X,Y) = -kg(X,JY)^2$. Putting this togheter in proposition 2.1, (vi) we get $g(X,JY) = 0$, thus H^+ and H^- are orthogonal, fact which is impossible since g is non-degenerate on H ■

It is now easy to construct to twistor structure on M, as well as the quaternionic Kähler structure on N. For elementary quaternionic-Kähler geometry we refer the reader to [15] and we note that all the constructions performed in the Riemannian case are available in the pseudo-Riemannian one.

Let m be a point of M and consider the linear span (in $End^-(H)$) of P_v, v in $\mathcal{V}$ and J. We got a 3-dimensional vector bundle, $\widetilde{Q}$, with fibers isomorphic at each point to $\mathfrak{sp}(1)$. Let now $\mathcal{L}_V^H$ be the horizontal projection of the Lie derivative (acting on $End^-(H)$) in the direction of V in $\mathcal{V}$. It is straightforward to verifiy that $\mathcal{L}_V^H$ leave $\widetilde{Q}$ invariant for all V in $\mathcal{V}$, hence $\widetilde{Q}$ projects on a rank 3 subbundle Q of $End^-(N)$. Since by (4.1) $\widetilde{Q}$ is $\overline{\nabla}$-parallel (and $\overline{\nabla}$ projects over the Levi-Civita connection of N) we obtain that Q is parallel for the Levi-Civita connection of the base space and therefore gives rise to a quaternionic-Kähler structure on (N,h). Moreover, by this construction, one easily see that M is the twistor space of this quaternionic-Kähler structure.

When M is complete and simply connected, so does N hence the last part of (iii) in theorem 1.1 follows by the fact that the quaternionic pseudo-hyperbolic space is the the single complete, simply connected, quaternionic-Kähler manifold of constant quaternionic curvature.

References

[1] G. BADITOIU, *Semi-Riemannian submersions with totally geodesic fibers*, eprint **arXiv:math**.DG/0005254

[2] G. BADITOIU, S. IANUS, *Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces*, Diff. Geom. Appl. **16** (2002), 79-94.

[3] M. BARROS, A. ROMERO *Indefinite Kähler manifolds*, Math. Ann. **261** (1982), 55-62.

[4] A. BESSE, *Einstein manifolds*, Springer-Verlag, New York, 1987.

[5] R. ESCOBALES, *Riemannian submersions with totally geodesic fibers*, J. Diff. Geom. **10** (1975), 253-276.

[6] R. ESCOBALES, *Riemannian submersions from complex projective spaces*, J.Diff.Geom. **13** (1978), 253-276.

[7] M. FALCITELLI, A. M. PASTORE *A note on almost Kähler and nearly Kähler submersions*, J. Geom. **69** (2000), 79-87.

[8] D. L. JOHNSON, *Kähler submersions and holomorphic connections*, J. Diff. Geom. **15** (1980), 71-79.

[9] S. KOBAYASHI, K. NOMIZU, *Foundations of differential geometry*, vol.1, Wiley, 1963.

[10] C. LEBRUN, *Fano manifolds, contact structures and quaternionic geometry*, International J. Math. **6** (1995), 419-437.

[11] M. MAGID, *Riemannian submersions from anti-de Sitter space with totally geodesic fibers*, J. Diff. Geom. **16**, (1981), 323-331.

[12] P. A. NAGY, *Rigidity of Riemannian foliations with complex leaves on Kähler manifolds*, eprint **arXiv:math. DG/0204046**.

[13] B. O' NEILL, *Semi-Riemannian geometry with application to relativity* , Academic Press, New York London 1983.

[14] PH. TONDEUR, *Geometry of Foliations*, Birkhäuser Verlag, 1997.

[15] S. SALAMON, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143-171.

[16] B. WATSON, *Almost hermitian submersions*, J. Diff. Geom. **11** (1976), 147-165.

Paul-Andi Nagy
Institut de Mathématiques
rue E. Argand 11, 2007 Neuchâtel
Switzerland
e-mail : Paul.Nagy@unine.ch