

Torsion in almost Kähler geometry *

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Abstract

We study almost Kähler manifolds whose curvature tensor satisfies the second curvature condition of Gray (shortly \mathcal{AK}_2). This condition is interpreted in terms of the first canonical Hermitian connection. Rather surprisingly, it turns out that it forces the torsion of this connection to be parallel in directions orthogonal to the Kähler nullity of the almost complex structure. We prove a local structure result for \mathcal{AK}_2 manifolds, showing that the basic pieces are manifolds with parallel torsion and special almost Kähler manifolds, a class generalizing, to some algebraic extent, the class of 4-dimensional \mathcal{AK}_2 -manifolds. In the case of parallel torsion, the Einstein condition and the reducibility of the canonical Hermitian connection is studied.

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1 Introduction

An almost Kähler manifold (shortly \mathcal{AK}) is a Riemannian manifold (M^{2n}, g) , together with a compatible almost complex structure J , such that the Kähler form $\omega = g(J\cdot, \cdot)$ is closed. Hence, almost Kähler geometry is nothing else than symplectic geometry with a preferred metric and complex structure. Since symplectic manifolds often arise in this way it is rather natural to ask under which conditions on the metric we get integrability of the almost complex J . In this direction, a famous conjecture of S.

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I. Golberg asserts that every compact, Einstein, almost Kähler manifold is, in fact, Kähler. At our present knowledge, this conjecture is still open. Nevertheless, there are a certain number of partial results, supporting this conjecture. First, let us note that Golberg conjecture is definitively not true with the compactness assumption removed. Now, K. Sekigawa proved in [20], that the Golberg conjecture is true when the scalar curvature is positive. The rest of known results, most of them enforcing or replacing the Einstein condition with some other natural curvature assumption are mainly in dimension 4.

We prove a local structure theorem concerning almost Kähler manifolds in the class \mathcal{AK}_2 .

Theorem 1.1 *Let (M^{2n}, g, J) be an almost Kähler manifold in the class \mathcal{AK}_2 . Let U be an open set where the Kähler nullity has constant rank. Then there is an open dense (with respect to the induced topology) set D in U such that around each point of D the manifold M is locally the Riemannian product of a almost Kähler manifold whose first canonical connection has parallel torsion and a special \mathcal{AK}_2 -manifold.*

The precise definition of special \mathcal{AK}_2 manifolds is given at the end of the section 4. They are those supporting \mathcal{AK}_2 structures for which the integral manifolds of the distribution orthogonal to the Kähler nullity are Kähler, with respect to the induced structure. Note, that algebraically (see definition 4.1 for details) this property is automatically satisfied in 4-dimensions.

Theorem 1.1 shows that the study of the torsion of an almost Kähler manifold of class \mathcal{AK}_2 reduces, in the local sense precised below to the study of the structure of the torsion of a special almost Kähler manifold.

The precise definition of the latter is given in section 4. Special almost Kähler manifolds can be thought as some kind of generalization of 4-dimensional \mathcal{AK}_2 manifolds, since they share the same algebraic properties of the first jet of the almost complex structure.

Concerning almost Kähler manifolds with parallel torsion, and in connection with the existence problem of Einstein, almost-Kähler metrics we are able to prove the following :

Theorem 1.2 *For every almost Kähler manifold with parallel torsion the holonomy of the canonical Hermitian connection is reducible, in the real sense. Furthermore, if such a manifold is Einstein, then it has to be Kähler.*

This shows that an Einstein \mathcal{AK}_2 manifold, if any, is locally the product of a Kähler Einstein manifold and an Einstein special \mathcal{AK}_2 -manifold. Note the difference with nearly-Kähler manifolds where many Einstein homogenous examples exist (see [15]).

Our paper is organised as follows. In section 2 we recall some general, well known facts of almost Kähler geometry. In section 3, the main technical ingredient of this paper is proved : using the first canonical Hermitian connection we give an interpretation of the second Gray condition on curvature in terms of the torsion of the last mentioned connection. Namely, we show that the associated (bundle valued) 1-form has to be closed. We study this condition using some standard methods and we prove

that the torsion of the canonical Hermitian connection has to be parallel in directions orthogonal to the Kähler nullity of the almost complex structure. Note that this result continues to hold in the more general case of quasi-Kähler manifolds satisfying the second Gray condition on curvature. Finally, in section 6, a proof of the theorem 2.1 is given, by using Sekigawas's formula (in fact its pointwise version developed in [3]) in the special case of parallel torsion.

2 Preliminaries

Let us consider an almost Hermitian manifold (M^{2n}, g, J) , that is a Riemannian manifold endowed with a compatible complex structure. We denote by ∇ the Levi-Civita connection of the Riemannian metric g . Consider now the tensor ∇J , the first derivative of the almost complex structure and recall that for all X in TM we have that $\nabla_X J$ is a skew-symmetric (with respect to g) endomorphism of TM , which anticommutes with J . The tensor ∇J can be used to distinguish various classes of almost Hermitian manifolds. For example, (M^{2n}, g, J) is quasi-Kähler iff

$$\nabla_{JX} J = -J \nabla_X J$$

for all X in TM . If $\omega = g(J\cdot, \cdot)$ denotes the Kähler form of the almost Hermitian structure (g, J) , we have an almost Kähler structure (\mathcal{AK} for short), iff $d\omega = 0$. We also recall the well known fact that almost Kähler manifolds are always quasi-Kähler.

The almost complex structure J defines a Hermitian structure if it is integrable, that is the Nijenhuis tensor N_J defined by

$$N_J(X, Y) = [JX, JY] - J[X, Y] - J[X, JY] - J[JX, Y]$$

for all vector fields X and Y on M vanishes. This is also equivalent to

$$\nabla_{JX} J = J \nabla_X J$$

whenever X is in TM . Therefore, an almost Kähler manifold which is also Hermitian must be Kähler.

In this paper we will deal mainly with almost Kähler (\mathcal{AK} for short)-manifolds, although we will authorize us short excursions to the quasi-Kähler class. We begin to recall some basic facts about the various notions of Ricci tensors. In the rest of this section (M^{2n}, g, J) will be a almost Kähler manifolds.

Let Ric be the Ricci tensor of the Riemannian metric g . We denote by Ric' and Ric'' the J -invariant resp. the J -anti-invariant part of the tensor Ric . Then the Ricci form is defined by

$$\rho = \langle Ric' J\cdot, \cdot \rangle .$$

We define the \star -Ricci form by

$$\rho^\star = \frac{1}{2} \sum_{i=1}^{2n} R(e_i, J e_i)$$

where $\{e_i, 1 \leq i \leq 2n\}$ is any local orthonormal basis in TM . Note that ρ^\star is not, in general, J -invariant. The \star -Ricci form is related to the Ricci form by

2.1

$$\rho^* - \rho = \frac{1}{2} \nabla^* \nabla \omega.$$

The (classical) proof of this fact consists in using the Weitzenböck formula for the harmonic 2-form ω . Taking the scalar product with ω we obtain :

$$s^* - s = \frac{1}{2} |\nabla J|^2$$

where the \star -scalar curvature is defined by $s^* = 2 \langle R(\omega), \omega \rangle$

We now come to properties of the curvature tensor of an almost Kähler manifold. Recall first that we have the decomposition :

$$\Lambda^2(M) = \Lambda^{1,1}(M) \oplus [[\Lambda^{2,0}(M)]]$$

and that $\Lambda^{1,1}(M) = \Lambda_0^{1,1}(M) \oplus \mathbb{R}.\omega$. Then the real vector bundle $[[\Lambda^{2,0}(M)]]$ has a complex structure \mathcal{J} defined by $(\mathcal{J}\alpha)(X, Y) = \alpha(JX, Y)$.

Let R be the curvature tensor of the metric g , with the convention that $R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$ for all vector fields X and Y on M . Let now \tilde{R} be the component of R acting trivially on $\Lambda^{1,1}(M)$. It is defined by :

$$\tilde{R}(X, Y, Z, U) = \frac{1}{4} (R(X, Y, Z, U) - R(JX, JY, Z, U) - R(X, Y, JZ, JU) + R(JX, JY, JZ, JU)).$$

Now, \tilde{R} decomposes further as $\tilde{R} = \tilde{R}' + \tilde{R}''$ where \tilde{R}' and \tilde{R}'' are the components of \tilde{R} commuting, resp. anticommuting with \mathcal{J} . Explicitely, we have :

$$\begin{aligned} \tilde{R}'(X, Y, Z, U) = \frac{1}{8} & \left(R(X, Y, Z, U) - R(JX, JY, Z, U) - R(X, Y, JZ, JU) + R(JX, JY, JZ, JU) \right. \\ & \left. - R(X, JY, Z, JU) - R(JX, Y, Z, JU) - R(X, JY, JZ, U) - R(JX, Y, JZ, U) \right) \end{aligned}$$

and

$$\begin{aligned} \tilde{R}''(X, Y, Z, U) = \frac{1}{8} & \left(R(X, Y, Z, U) - R(JX, JY, Z, U) - R(X, Y, JZ, JU) + R(JX, JY, JZ, JU) \right. \\ & \left. + R(X, JY, Z, JU) + R(JX, Y, Z, JU) + R(X, JY, JZ, U) + R(JX, Y, JZ, U) \right) \end{aligned}$$

for all X, Y, Z, U in TM . An important information about this tensors is given by an identity of A. Gray from [13], page 604, Cor.4.3 :

$$\mathbf{2.2} \quad \tilde{R}''(X, Y, Z, U) = -\frac{1}{4} \langle (\nabla_X J)Y - (\nabla_Y J)X, (\nabla_Z J)U - (\nabla_U J)Z \rangle .$$

We will now give an important formula, which can be interpreted as an obstruction to the existence of almost Kähler, non-Kähler structures. This is a corrected version of the formula in [3], where it was accidentally asserted that \tilde{R}' is given by the expression in formula (2.2).

Proposition 2.1 *Let (M^{2n}, g, J) be an almost Kähler manifold. Then the following holds :*

$$\begin{aligned} \Delta(s^* - s) = & -4\delta(J\delta(JRic'')) + 8\delta(\langle \rho^*, \nabla.\omega \rangle) + 2|Ric''|^2 \\ & + 4 \langle \rho, \Phi - \nabla^* \nabla \omega \rangle - |\Phi|^2 - |\nabla^* \nabla \omega|^2 - 8|\tilde{R}'|^2. \end{aligned}$$

Here, the semi-positive 2-form Φ is defined by

$$\Phi(X, Y) = \langle \nabla_{JX}\omega, \nabla_Y\omega \rangle .$$

Proof :

The proof is exactly that the one given in [3], where one has to make use of (2.2), the correct "translation" of the identity of Gray. ■

3 Gray's curvature conditions

This section is dedicated to interpret some of the well known conditions on the curvature of a almost Hermitian manifold in terms of the torsion of the first canonical Hermitian connection. We begin by recalling how one can distinguish several classes of almost Hermitian manifolds by "the degree of resemblance" of their Riemannian curvature tensor with the curvature tensor of a Kähler manifold.

Let (M^{2n}, g, J) be an almost Hermitian manifold. Let R be the Riemannian curvature tensor of the metric g . Then the following classes of almost Hermitian manifolds appear in a natural way [13]:

$$(G_1) : R(X, Y, JZ, JU) = R(X, Y, Z, U)$$

$$(G_2) : R(X, Y, Z, U) - R(JX, JY, Z, U) = R(JX, Y, JZ, U) + R(JX, Y, Z, JU)$$

$$(G_3) : R(JX, JY, JZ, JU) = R(X, Y, Z, U)$$

Using the first Bianchi identity it is a simple exercise to see that $G_1 \Rightarrow G_2 \Rightarrow G_3$. It is also clear that a Kähler structure satisfies all the three conditions. Let us set now some notations.

Following [13], let \mathcal{AK} be the class of almost Kähler manifolds. Then the class $\mathcal{AK}_i, 1 \leq i \leq 3$ contains those almost Kähler manifolds whose curvature tensor satisfies the condition (G_i) . Obviously, we have the inclusions :

$$\mathcal{AK}_1 \subseteq \mathcal{AK}_2 \subseteq \mathcal{AK}_3.$$

Note that it was shown in [12] that locally $\mathcal{AK}_1 = \mathcal{K}$, where \mathcal{K} denotes the class of Kähler manifolds. The other inclusions are strict in dimensions $2n \geq 6$, as shows the examples of [9], multiplied by Kähler manifolds. In the same spirit, the class $\mathcal{AH}_i, 1 \leq i \leq 3$ contains those almost Hermitian manifolds whose Riemannian curvature tensor satisfies condition (G_i) .

Let us consider now the first canonical connection of the almost Hermitian manifold (M^{2n}, g, J) to be defined by :

$$\bar{\nabla}_X Y = \nabla_X Y + \eta_X Y$$

whenever X, Y are vector fields on M , where ∇ is the Levi-Civita connection of g and where, to save space, we setted $\eta_X Y = \frac{1}{2}(\nabla_X J)JY$. We obtain a metric Hermitian connection on M , that is $\bar{\nabla}g = 0$ and $\bar{\nabla}J = 0$. Recall that for each X in TM η_X is a skew-symmetric, J -anticommuting endomorphism of TM .

The torsion tensor of the canonical Hermitian canonical connection, to be denoted by T is given by

$$T_X Y = \eta_X Y - \eta_Y X$$

for all X, Y in TM . Then by the **torsion** of the almost Hermitian manifold (M^{2n}, g, J) we will mean simply the torsion tensor of the canonical Hermitian connection.

For the almost Hermitian (M^{2n}, g, J) be almost Kähler one requires that the Kähler form $\omega(X, Y) = \langle JX, Y \rangle$ be closed. In our notations, this is equivalent to have

$$\mathbf{3.1} \quad \langle T_X Y, Z \rangle = - \langle \eta_Z X, Y \rangle .$$

In the almost Kähler context, this relation will be used almost implicitly in the rest of this paper.

Let $\bar{R}(X, Y) = -[\bar{\nabla}_X, \bar{\nabla}_Y] + \bar{\nabla}_{[X, Y]}$ be the curvature tensor of the connection $\bar{\nabla}$. Now a standard calculation involving the definitions yields to

$$\mathbf{3.2} \quad \bar{R}(X, Y)Z = R(X, Y)Z + [\eta_X, \eta_Y]Z - [d_{\bar{\nabla}}u(X, Y)]Z$$

where

$$[d_{\bar{\nabla}}u(X, Y)]Z = (\bar{\nabla}_X \eta)(Y, Z) - (\bar{\nabla}_Y \eta)(X, Z) + \eta_{T_X Y} Z$$

for all vector fields X, Y, Z on M . Note that $d_{\bar{\nabla}}u(X, Y)$ is a J -anticommuting endomorphism of TM , whenever X, Y are tangent vectors to M .

Remark 3.1 *In the formula (3.1), the notation u stands for the tensor η , considered as a 1-form with values in the bundle $\Omega^2(M)$. Then $d_{\bar{\nabla}}$, with its expression given below, is the twisted differential acting on twisted one forms, when considering the tangent bundle of M endowed with the connection $\bar{\nabla}$. Since our discussion is intended to be self-contained and at the elementary level, we will keep things at the level of the notation after (3.1).*

The fact that $\bar{\nabla}$ is a Hermitian connection implies that $\bar{R}(X, Y, JZ, JU) = \bar{R}(X, Y, Z, U)$. Using this in formula (2.1), together with the skew-symmetry of the J -anticommuting endomorphism η_X gives us :

$$\mathbf{3.3} \quad R(X, Y, Z, U) - R(X, Y, JZ, JU) = 2 \left[(d_{\bar{\nabla}}u)(X, Y) \right] (Z, U).$$

Using the symmetry property of the Riemannian curvature we also deduce that

$$\mathbf{3.4} \quad R(X, Y, Z, U) - R(JX, JY, Z, U) = 2 \left[(d_{\bar{\nabla}}u)(Z, U) \right] (X, Y).$$

Lemma 3.1 *The almost Hermitian manifold (M^{2n}, g, J) satisfies the condition (G_3) iff*

$$\mathbf{3.5} \quad \left[(d_{\bar{\nabla}}u)(Z, U) \right] (X, Y) = \left[(d_{\bar{\nabla}}u)(X, Y) \right] (Z, U).$$

Moreover, we have $(d_{\bar{\nabla}}u)(JX, JY) + (d_{\bar{\nabla}}u)(X, Y) = 0$, hence $d_{\bar{\nabla}}u$ defines a symmetric endomorphism of $\Lambda_-^2(M)$.

Proof :

We change Z and U in JZ and JU respectively in (3.3) and take the sum with (3.2).

Using (G_3) , we get $\left[(d_{\bar{\nabla}}u)(JZ, JU) \right] (X, Y) = - \left[(d_{\bar{\nabla}}u)(X, Y) \right] (Z, U)$. To get (3.4) we change again Z in JZ and U in JU and use that $du_{\bar{\nabla}}(X, Y)$ is J -anticommuting. The rest is straightforward. ■

This simple observation has a number of usefull consequences related to the algebraic symmetries of the tensor \bar{R} .

Corollary 3.1 *Let (M^{2n}, g, J) be a quasi-Kähler manifold in the class \mathcal{AH}_3 and let X, Y, Z, U be vector fields on M . The following holds :*

(i)

$$\overline{R}(X, Y, Z, U) - \overline{R}(Z, U, X, Y) = \langle [\eta_X, \eta_Y]Z, U \rangle - \langle [\eta_Z, \eta_U]X, Y \rangle .$$

(ii) $\overline{R}(JX, JY), Z, U) = \overline{R}(X, Y, Z, U)$.

Proof :

Property (i) follows immediately from (3.1), the symmetry of Riemannian curvature operator and lemma 3.1. Since \overline{R} is a Hermitian connection we have $\overline{R}(Z, U, JX, JY) = \overline{R}(Z, U, X, Y)$, hence (ii) follows from (i) and the quasi-Kähler condition on the tensor η . ■

Note that the previous corollary is well known for nearly-Kähler manifolds (see [14] for instance). Also note that property (i) holds in fact for any almost Hermitian manifold in the class \mathcal{AH}_3 .

We can have now a clearer understanding of the tensor \tilde{R}' appearing in the Sekigawa's formula (see proposition 2.1) in terms of the torsion of the canonical Hermitian connection.

Corollary 3.2 *Suppose that (M^{2n}, g, J) belongs to the class \mathcal{AK}_3 . Then :*

$$\tilde{R}'(X, Y, Z, U) = \langle (\overline{\nabla}_X \eta)(Y, Z) - (\overline{\nabla}_Y \eta)(X, Z), U \rangle$$

whenever X, Y, Z, U belongs to TM .

Proof :

This is a simple computation involving the definition of the tensor \tilde{R}' (see section 2) and lemma 3.1. It will be left to the reader. ■

With this preliminaries in mind, it turns out that quasi-Kähler manifolds in the class \mathcal{AH}_2 have a particularly nice description in terms of the torsion.

Proposition 3.1 *Let (M^{2n}, g, J) be a quasi-Kähler manifold. Then M satisfies condition (G_2) iff*

$$3.6 \quad (\overline{\nabla}_X \eta)(Y, Z) = (\overline{\nabla}_Y \eta)(X, Z)$$

whenever X, Y, Z are in TM . In particular :

$$3.7 \quad (\overline{\nabla}_{JX} \eta)(JY, Z) + (\overline{\nabla}_X \eta)(Y, Z) = 0.$$

Proof :

Using (3.2) and (3.3) it is straightforward to see that condition (G_2) is equivalent with

$$3.8 \quad \left[(d_{\overline{\nabla}} u)(Z, U) \right] (X, Y) = \left[(d_{\overline{\nabla}} u)(JX, Y) \right] (JZ, U)$$

Expanding the right hand side of this equation and taking into account that (M^{2n}, g, J) is quasi-Kähler gives

$$\begin{aligned}
& \left[(d_{\overline{\nabla}}u)(Z, U) \right] (X, Y) = \\
\mathbf{3.9} \quad & \langle (\overline{\nabla}_{JX}\eta)(JY, Z), U \rangle + \langle (\overline{\nabla}_Y\eta)(X, Z), U \rangle + \langle \eta_{T_X Y} Z, U \rangle = \\
& \langle (\overline{\nabla}_{JX}\eta)(JY, Z) + (\overline{\nabla}_X\eta)(Y, Z), U \rangle - \left[(d_{\overline{\nabla}}u)(X, Y) \right] (Z, U) + 2 \langle \eta_{T_X Y} Z, U \rangle .
\end{aligned}$$

But M equally satisfies condition (G_3) , hence using (3.4) in (3.9) we obtain

$$\begin{aligned}
\mathbf{3.10} \quad & 2 \left[(d_{\overline{\nabla}}u)(X, Y) \right] (Z, U) - 2 \langle \eta_{T_X Y} Z, U \rangle = \\
& \langle (\overline{\nabla}_{JX}\eta)(JY, Z), U \rangle + \langle (\overline{\nabla}_X\eta)(Y, Z), U \rangle .
\end{aligned}$$

But the right side of this equation, viewed as a tensor in X and Y is J -invariant, whilst the right hand side is J -antiinvariant. Therefore, both have to vanish and this finishes the proof ■

Remark 3.2 *From the previous proposition we deduce that quasi-Kähler, class \mathcal{AH}_2 -manifolds have to satisfy the algebraic constraint :*

$$\langle \eta_{T_X Y} U, W \rangle = \langle \eta_{T_V W} X, Y \rangle .$$

Indeed, this follows immediately from (3.6) and lemma 3.1. Note also that this relation is automatically satisfied in both nearly Kähler and almost Kähler cases.

Using 3.1 we obtain that for a quasi-Kähler manifold in the class \mathcal{AH}_2 , the difference between the curvature of the canonical Hermitian connection and the Riemannian curvature tensor is simply expressed by :

$$\mathbf{3.11} \quad \overline{R}(X, Y)Z = R(X, Y)Z + [\eta_X, \eta_Y]Z - \eta_{T_X Y} Z$$

for all vector fields X, Y, Z on M . Note that this difference behaves as the torsion of the canonical Hermitian was parallel.

We will exploit the second order version of (1.6) as follows.

Lemma 3.2 *Let (M^{2n}, g, J) be a quasi-Kähler manifold in the class \mathcal{AH}_2 . Then for all vector fields X, Y, Z on M we have :*

$$\mathbf{3.12} \quad \sigma_{X, Y, Z} \left([\overline{R}(X, Y), \eta_Z] - \eta_{\overline{R}(X, Y)Z} \right) + \sigma_{X, Y, Z} (\overline{\nabla}_{T_X Y} \eta)(Z, \cdot) = 0$$

where σ denotes the cyclic sum.

Proof :

Starting from $(\overline{\nabla}_Y \eta)(Z, U) = (\overline{\nabla}_Z \eta)(Y, U)$ we obtain after derivating in the direction of the vector field X that : $(\overline{\nabla}_{X, Y}^2 \eta)(Z, U) = (\overline{\nabla}_{X, Z}^2 \eta)(Y, U)$. It is elementary to get then :

$$\sigma_{X, Y, Z} \left[(\overline{\nabla}_{X, Y}^2 \eta)(Z, U) - (\overline{\nabla}_{Y, X}^2 \eta)(Z, U) \right] = 0.$$

Using the Ricci identity for the connection with torsion $\overline{\nabla}$ (see [8] for instance) we arrive at :

$$\sigma_{X, Y, Z} (\overline{R}(X, Y)\eta)(Z, U) + \sigma_{X, Y, Z} (\overline{\nabla}_{T_X Y} \eta)(Z, U) = 0.$$

We end the proof by recalling that the action of the curvature on η is explicitly given by :

$$(\overline{R}(X, Y)\eta)(Z, \cdot) = [\overline{R}(X, Y), \eta_Z] - \eta_{\overline{R}(X, Y)Z}.$$

■

Proposition 3.2 *Let (M^{2n}, g, J) be a quasi-Kähler manifold in the class \mathcal{AH}_2 . Then we have :*

$$\mathbf{3.13} \quad \overline{\nabla}_{T_X Y} \eta = 0.$$

Proof :

Replacing in (3.12) X, Y, Z by JX, JY, JZ respectively we obtain

$$\sigma_{X, Y, Z} \left([\overline{R}(JX, JY), \eta_{JZ}] - \eta_{\overline{R}(JX, JY)JZ} \right) + \sigma_{X, Y, Z} (\overline{\nabla}_{T_{JX} JY} \eta)(JZ, \cdot) = 0$$

Now, $\eta(JU, \cdot) = -J\eta(U, \cdot)$ for all U in TM , and it follows that

$$-J\sigma_{X, Y, Z} \left([\overline{R}(JX, JY), \eta_Z] - \eta_{\overline{R}(JX, JY)Z} \right) + J\sigma_{X, Y, Z} (\overline{\nabla}_{T_X Y} \eta)(Z, \cdot) = 0.$$

Using now corollary 3.1, (ii) and lemma 3.3 we get $\sigma_{X, Y, Z} (\overline{\nabla}_{T_X Y} \eta)(Z, \cdot) = 0$, or in expanded version :

$$\mathbf{3.14} \quad (\overline{\nabla}_{T_X Y} \eta)(Z, \cdot) + (\overline{\nabla}_{T_Y Z} \eta)(X, \cdot) + (\overline{\nabla}_{T_Z X} \eta)(Y, \cdot) = 0.$$

It is clear that at Z fixed, the first term of (3.14) is J -anti-invariant. Now, $(\overline{\nabla}_{T_{JY} Z} \eta)(JX, \cdot) = -(\overline{\nabla}_{J T_Y Z} \eta)(JX, \cdot) = (\overline{\nabla}_{T_Y Z} \eta)(X, \cdot)$ by (3.7), hence the last two terms of (3.15) are J -invariant. This clearly ends the proof of the proposition. ■

4 A first decomposition result

In this section we will analyse some important geometric consequences of the proposition 3.2 in the last section in the particular context of almost Kähler geometry. Our main object of study will be an almost Kähler manifold (M^{2n}, g, J) belonging to the class \mathcal{AK}_2 .

Let us first set a notational convention, to be used intensively in the present and the next section and intended to improve presentation. If E and F are vector subbundles of TM and Q is a tensor of type $(2, 1)$, we will denote by $Q(E, F)$ (or $Q_E F$) the subbundle of TM generated by elements of the form $Q(u, v)$ where u belongs to E and v is in F . We will also denote by $\langle E, F \rangle$ the product of two generic elements of E and F respectively.

An important object associated with an almost Kähler manifold is its **Kähler nullity**. This is the vector bundle H over M defined at a point m of M by $H_m = \{v \in T_m M : \nabla_v J = 0\}$. We also define \mathcal{V} to be the orthogonal complement of H in TM . It is easy to see that at each point of M we have that \mathcal{V} is generated by elements of the form $T_X Y, X, Y$ in TM , in other words

$$\mathcal{V} = T(TM, TM).$$

Hence, we have a orthogonal, J -invariant decomposition

4.1

$$TM = \mathcal{V} \oplus H.$$

Note that, a priori, H need not to have constant rank over M . However, this is true locally, in the following sense. Call a point m of M **regular** if the rank of η attains a local maximum at m . Using standard continuity arguments, it follows that around each regular point, the rank of η , and hence that of H is constant in some open subset. It is also easy to see that the set of regular point is dense i in M , provided that the manifold is connected. As we are concerned with the local (in some neighbourhood of a regular point) structure of \mathcal{AK}_2 we can assume, without loss of generality, that H has constant rank over M . This assumption will be made in the whole rest of this section.

Let us examine now some elementary properties of the decomposition (4.1).

Lemma 4.1 (i) $(\overline{\nabla}_U T)(U_1, U_2)$ belongs to H for all vector fields U, U_1, U_2 on TM .

(ii) $\overline{\nabla}_V X$ belongs to H for all V in \mathcal{V} and X in H .

(iii) $\overline{\nabla}_V W$ belongs to \mathcal{V} if V, W are in \mathcal{V} .

Proof :

(i) Start from the relation $\langle T_X Y, T_Z U \rangle = - \langle \eta_{T_X Y} Z, U \rangle$. Derivating in the direction of U_1 we get

$$\begin{aligned} & \langle (\overline{\nabla}_{U_1} T)(X, Y), T_Z U \rangle + \langle T_X Y, (\overline{\nabla}_{U_1} T)(Z, U) \rangle = \\ & - \langle (\overline{\nabla}_{U_1} \eta)(T_X Y, Z) \rangle - \langle \eta_{(\overline{\nabla}_{U_1} T)(X, Y)} Z, U \rangle. \end{aligned}$$

But the first term of the right hand side vanishes by (3.6) and (3.13). Moreover, by (3.5) $\langle (\overline{\nabla}_{U_1} T)(X, Y), T_Z U \rangle = - \langle \eta_{(\overline{\nabla}_{U_1} T)(X, Y)} Z, U \rangle$ hence it follows that $\langle T_X Y, (\overline{\nabla}_{U_1} T)(Z, U) \rangle = 0$ and since \mathcal{V} is generated by $\{T_X Y : X, Y \in TM\}$ the proof is finished.

(ii) We know by (3.13) that $(\overline{\nabla}_V \eta)(X, U) = 0$ for all U in TM . Since $\eta_X = 0$ this gives $\eta_{\overline{\nabla}_V X} U = 0$ and the proof is finished. Now, (iii) is a straightforward consequence of (ii). ■

As an immediate consequence we obtain our first information concerning the nature of the distributions \mathcal{V} and H .

Corollary 4.1 Both distributions \mathcal{V} and H are integrable.

Proof :

The integrability of \mathcal{V} follows directly from lemma 4.1 (iii) and the fact that $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$. To prove (ii) we use that $(\overline{\nabla}_X \eta)(Y, U) = (\overline{\nabla}_Y \eta)(X, U)$ for all X, Y in H and U in TM (see proposition 3.1). Since $T(H, H) = 0$, this is equivalent to $\eta_{[X, Y]} U = 0$ and this implies readily that $[X, Y]$ belongs to H . ■

Note that in the context of quasi-Kähler \mathcal{AH}_2 manifolds it is already known [13] that H is an integrable distribution, over each open subset of M where it has constant rank.

Lemma 4.2 For all (vector fields V, W belonging to \mathcal{V} and X, Y in H respectively we have :

(i) $\overline{R}(V, W)\eta = 0$.

(ii) $[\overline{R}(X, Y), \eta_V] = \eta_{\beta_V(X, Y)}$, where $\beta_V(X, Y) = \eta_{m_V Y}X - \eta_{m_V X}Y$.

Proof :

(i) We know that $\overline{\nabla}\eta$ vanishes in vertical directions (cf. 3.13). By derivation, and taking into account lemma 4.1, (iii) the result follows.

(ii) We use (3.12) actualized by (3.13). We have :

$$[\overline{R}(X, Y), \eta_V] + [\overline{R}(Y, V), \eta_X] + [\overline{R}(V, X), \eta_Y] = \eta_{\overline{R}(X, Y)V + \overline{R}(Y, V)X + \overline{R}(V, X)Y}.$$

Now the last two terms of the first member are clearly vanishing and the use of the first Bianchi identity for $\overline{\nabla}$ yields after a short computation to the claimed result. ■

We are now going to obtain a first decomposition of the vector bundle \mathcal{V} having good algebraic properties with respect to the torsion tensor T . Define a subbundle \mathcal{V}_0 of \mathcal{V} by setting

$$\mathcal{V}_0 = T(\mathcal{V}, \mathcal{V})$$

and let \mathcal{V}_1 be its orthogonal complement in \mathcal{V} . In fact, \mathcal{V}_1 can be considered as the Kähler nullity of the foliation induced by \mathcal{V} with respect to the induced almost Kähler structure. More precisely, define a tensor

$$\hat{\eta} : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \text{ by } \hat{\eta}_v w = (\eta_v w)_\mathcal{V}.$$

Then we have $\mathcal{V}_1 = \{v \in \mathcal{V} : \hat{\eta}_v \mathcal{V} = 0\}$. In other words, we have $\eta_{\mathcal{V}_1} \mathcal{V} \subseteq H$ and this implies that $T(\mathcal{V}_1, \mathcal{V}_1) = 0$. Note that $\hat{\eta}$ completely determines the torsion over \mathcal{V} , that is $\hat{\eta}_v w - \hat{\eta}_w v = T(v, w)$ for all v, w in \mathcal{V} . This follows from $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$.

In the subsequent, we will assume that the subbundle \mathcal{V}_0 has constant rank over M . As our study is purely local and we have already assumed that H has constant rank over M , there is no loss of generality since \mathcal{V}_0 has constant rank around each point of some open dense subset of M .

Remark 4.1 *In the subsequent we will treat the distribution \mathcal{V} as it were an almost Kähler manifold with parallel torsion and \mathcal{V}_1 its Kähler nullity. This approach is motivated by the simple observation that the integral manifolds of \mathcal{V} with respect to the induced metric and almost complex structure are almost Kähler manifolds with parallel torsion. Moreover, one can see that the first canonical Hermitian connection of such an integral manifold, coincides with the restriction of $\overline{\nabla}$.*

The point of departure of our study is the following :

Lemma 4.3 *The orthogonal decomposition $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ is J -invariant and $\overline{\nabla}$ -parallel inside \mathcal{V} .*

Proof :

From (3.13) we get that $\overline{\nabla}_V T = 0$ for all V in \mathcal{V} . A routine use of lemma 4.1, (iii) yields now to the parallelism of \mathcal{V}_0 and hence to that of \mathcal{V}_1 inside \mathcal{V} . ■

In the following lemma we show that the existence of such a decomposition generates strong algebraic restrictions involving the tensors T and $\hat{\eta}$.

Lemma 4.4 (i) Suppose that we have an orthogonal, J -invariant, decomposition $\mathcal{V} = D_1 \oplus D_2$ where the distributions D_1 and D_2 are J -invariant and $\bar{\nabla}$ -parallel inside \mathcal{V} . Then we have :

$$\bar{R}(v_1, w_1, w, w_2) = - \langle T_{w_1} w, \hat{\eta}_{w_2} v_1 \rangle - \langle T_w v_1, \hat{\eta}_{w_2} v_1 \rangle$$

for all v_1, w_1 in D_1 , w in \mathcal{V} and w_2 in D_2 .

(ii) under the assumptions in (i) we have $\hat{\eta}_{D_2} T(D_1, D_1) = 0$.

(iii) $\bar{R}(v, w, v_1, w_1) = 0$ for all v, w in \mathcal{V}_0 and v_1, w_1 in \mathcal{V}_1 .

(iv) for all $v_i, 1 \leq i \leq 4$ in \mathcal{V} we have :

$$\bar{R}(v_1, v_2, v_3, v_4) - \bar{R}(v_3, v_4, v_1, v_2) = \langle [\hat{\eta}_{v_1}, \hat{\eta}_{v_2}] v_3, v_4 \rangle - \langle [\hat{\eta}_{v_3}, \hat{\eta}_{v_4}] v_1, v_2 \rangle .$$

Proof :

We will prove (i) and (ii) in the same time. Using the first Bianchi identity for the connexion $\bar{\nabla}$, we get :

$$\begin{aligned} & \bar{R}(v_1, w_1, w, w_2) + \bar{R}(w_1, w, v_1, w_2) + \bar{R}(w, v_1, w_1, w_2) + \\ & \langle T_{v_1} w_1, \eta_{w_2} w \rangle + \langle T_{w_1} w, \eta_{w_2} v_1 \rangle + \langle T_w v_1, \eta_{w_2} w_1 \rangle = 0. \end{aligned}$$

Now the second and the third term below vanish since D_1, D_2 are $\bar{\nabla}$ -parallel inside \mathcal{V} . Using that $\bar{R}(Jv_1, Jw_1, w, w_2) = \bar{R}(v_1, w_1, w, w_2)$ and the J -invariance properties of the tensor T it follows easily that $\langle T_{v_1} w_1, \eta_{w_2} w \rangle = 0$. Hence $\langle T_{v_1} w_1, \hat{\eta}_{w_2} w \rangle = 0$ and since w in \mathcal{V} was chosen arbitrary we get (ii). The proof of (i) is now straightforward.

To prove (iii) we take $D_1 = \mathcal{V}_0, D_2 = \mathcal{V}_1$ in (i) and use that \mathcal{V}_1 is the Kähler nullity of $\hat{\eta}$. Finally, for the proof of (iv) we use corollary 3.1, (i) and the fact that the tensor η^H defined by $\eta_v^H w = (\eta_v w)_H$ is symmetric for all v, w in \mathcal{V} (this is a consequence of the fact that the torsion is concentrated in \mathcal{V}). ■

We are now able to prove our first decomposition result as follows.

Proposition 4.1 The subbundle \mathcal{V}_0 admits an orthogonal, J -invariant decomposition

$$\mathcal{V}_0 = W_1 \oplus W_2$$

which is $\bar{\nabla}$ -parallel inside \mathcal{V} and has the following algebraic properties :

(i) $W_1 = T(W_1, W_1)$

(ii) $\hat{\eta}_{W_1} W_2 = \hat{\eta}_{W_2} W_1 = 0$

(iii) $\hat{\eta}_{W_2} W_2 \subseteq \mathcal{V}_1$.

(iv) $\hat{\eta}_{W_2} \mathcal{V}_1 = W_2$.

Proof :

We define $W_1 = T(\mathcal{V}_0, \mathcal{V}_0)$ and let W_2 be the orthogonal complement of W_1 in \mathcal{V}_0 . It is clear that $W_i, i = 1, 2$ are J -invariant and $\bar{\nabla}$ -parallel inside \mathcal{V} . Hence, we have a $\bar{\nabla}$ -parallel decomposition (inside \mathcal{V})

$$\mathcal{V} = W_1 \oplus (W_2 \oplus \mathcal{V}_1).$$

Using lemma 4.4, (ii) with $D_1 = W_1, D_2 = W_2 \oplus \mathcal{V}_1$ we get $\hat{\eta}_{W_1} T(W_2, \mathcal{V}_1) = 0$ hence

4.2

$$\hat{\eta}_{W_1} \hat{\eta}_{W_2} \mathcal{V}_1 = 0$$

since $\hat{\eta}_{\mathcal{V}_1} \mathcal{V} = 0$. It follows that

$$\langle \hat{\eta}_{W_2} \mathcal{V}_1, \hat{\eta}_{W_1} W_2 \rangle = 0.$$

Now, the definition of W_2 implies that $\langle W_2, T(\mathcal{V}_0, \mathcal{V}_0) \rangle = 0$ and the use of the almost Kähler condition (3.1) gives :

4.3

$$\hat{\eta}_{W_2} \mathcal{V}_0 \subseteq \mathcal{V}_1.$$

In particular, $\hat{\eta}_{W_2} W_1 \subseteq \mathcal{V}_1$. Then $\langle \hat{\eta}_{W_2} \mathcal{V}_1, \hat{\eta}_{W_2} W_1 \rangle = 0$ since the vanishing of the torsion on \mathcal{V}_1 implies that $\hat{\eta}_{W_2} \mathcal{V}_1$ is orthogonal to \mathcal{V}_1 . We deduce that $\langle \hat{\eta}_{W_2} \mathcal{V}_1, T(W_1, W_2) \rangle = 0$, in other words $\hat{\eta}_{W_2} T(W_1, W_2)$ is orthogonal to \mathcal{V}_1 . But $T(W_1, W_2) \subseteq W_1$ by the definition of W_1 and we saw that $\hat{\eta}_{W_2} W_1 \subseteq \mathcal{V}_1$. It follows that

$$\hat{\eta}_{W_2} T(W_1, W_2) = 0.$$

Consider now the orthogonal, J -invariant and $\bar{\nabla}$ -parallel (inside \mathcal{V}) decomposition $\mathcal{V} = W_2 \oplus (W_1 \oplus \mathcal{V}_1)$. Using again lemma 4.4, (ii) we get

$$\hat{\eta}_{W_2} T(W_1, W_1) = 0.$$

Now, since $T(W_2, W_2) = 0$ (this follows immediately from (4.3)) and by the definition of W_1 we have that W_1 is generated by $T(W_1, W_1)$ and $T(W_2, W_1)$ and by the previous discussion we obtain :

$$\hat{\eta}_{W_2} W_1 = 0.$$

Therefore, the second half of property (ii) is now proven .

Now, $\hat{\eta}_{W_2} \mathcal{V}_1$ is orthogonal to W_1 , but we also know that it is orthogonal to \mathcal{V}_1 as $T(\mathcal{V}_1, \mathcal{V}_1) = 0$. It follows that $\hat{\eta}_{W_2} \mathcal{V}_1 \subseteq W_2$. Let define now $E = \hat{\eta}_{W_2} \mathcal{V}_1$ and let F be the orthogonal complement of E in W_2 . Then $\hat{\eta}_{W_2} F$ is orthogonal to \mathcal{V}_1 and then by (4.3) we obtain that $\hat{\eta}_{W_2} F = 0$. Since $T(W_2, W_2) = 0$ we also have $\hat{\eta}_F W_2 = 0$. But $\hat{\eta}_F \mathcal{V}_1 \subseteq E \subseteq W_2$ and then $\hat{\eta}_F \mathcal{V}_1 = 0$. Or F is contained in W_2 hence $\hat{\eta}_F W_1 = 0$. We showed that $\hat{\eta}_F \mathcal{V} = 0$ and since F is contained in \mathcal{V}_0 it has to vanish.

We get that $\hat{\eta}_{W_2} \mathcal{V}_1 = W_2$, proving the property (iv). Now using (4.2) we obtain that $\hat{\eta}_{W_1} W_2 = 0$ finishing the proof of (ii). Then $T(W_1, W_2) = 0$ and this implies that $W_1 = T(W_1, W_1)$. Then (i) is also proved, and (iii) is an easy consequence of (ii) and of vanishing of the torsion on W_2 . ■

We end this section with the following definition.

Definition 4.1 *Let (M^{2n}, g, J) be in the class \mathcal{AK}_2 . It is said to be **special** iff H is of constant rank over M and $(\nabla_{\mathcal{V}} J)\mathcal{V} = H$.*

From an intuitive point of view, special \mathcal{AK}_2 manifolds are those for which the integral manifolds of the distribution \mathcal{V} , the orthogonal of the Kähler nullity are Kähler. with respect to the induced structure. Furthermore, the equality required in the definition forbids product with Kähler manifolds.

Remark 4.2 (i) Every 4-dimensional manifold in the class \mathcal{AK}_2 is special on the open set where its Nijenhuis tensor does not vanish.

(ii) If (M^{2n}, g, J) is a special \mathcal{AK}_2 -manifold then it follows directly from the definition that $T(\mathcal{V}, \mathcal{V}) = 0$. We also have

$$(\nabla_{\mathcal{V}}J)H = \mathcal{V}.$$

Indeed, if E is the orthogonal complement of $(\nabla_{\mathcal{V}}J)H$ in \mathcal{V} then $\eta_{\mathcal{V}}F = 0$. The vanishing of the torsion on \mathcal{V} implies then $\eta_F\mathcal{V} = 0$. In other words, $\eta_F H$ is orthogonal to \mathcal{V} and then it has to vanish. We showed that F is in fact contained in the Kähler nullity of (g, J) hence $F = 0$ and our assertion follows.

5 Curvature properties

In this section we will examine the curvature tensor of a local \mathcal{AK}_2 -manifold. After proving some general properties we will show how the algebraic-geometric properties of \overline{R} can be used to obtain more information about the algebraic nature of the decomposition given in proposition 4.1. Finally, this study will lead to the proof of theorem 1.1, which is given at the end of this section.

Throughout this section (M^{2n}, g, J) will be an almost Kähler manifold in the class \mathcal{AK}_2 . All the notations in the previous section will be used without further comment.

Lemma 5.1 Let $V_i, 1 \leq i \leq 3$ be in \mathcal{V} and X in H . We have :

- (i) $\overline{R}(V_1, V_2, V_3, X) = 0$
- (ii) $\overline{R}(X, V_1, V_2, V_3) = - \langle [\eta_{V_2}, \eta_{V_3}]X, V_1 \rangle$
- (iii) $(\overline{\nabla}_V \overline{R})(X, V_1, V_2, V_3) = 0$.

Proof :

(i) follows directly from lemma 4.1, (iii) and the integrability of \mathcal{V} . To obtain (ii) one uses the symmetry property of corollary 3.1, (i). Finally, (iii) follows by derivating (ii) and taking into account that $\overline{\nabla}_V X$ belongs to H for all X in H and V in \mathcal{V} and the fact that $\overline{\nabla}_V \eta = 0$. ■

We will now use the second Bianchi identity for the canonical Hermitian connection in order to get more information about the algebraic properties of ∇J with respect to the decomposition (4.1).

Proposition 5.1 Let $X, V_i, 1 \leq i \leq 4$ be vector fields on H and \mathcal{V} respectively. We have :

- (i)
- 5.1** $\overline{R}(\eta_{V_2}X, V_1, V_3, V_4) - \overline{R}(\eta_{V_1}X, V_2, V_3, V_4) = - \langle [\eta_{V_3}, \eta_{V_4}]X, T_{V_1}V_2 \rangle .$
- (ii) $(\overline{\nabla}_X \overline{R})(V_1, V_2, V_3, V_4) = 0$.

Proof :

Using the second Bianchi identity we obtain

$$\begin{aligned} & (\overline{\nabla}_X \overline{R})(V_1, V_2, V_3, V_4) + (\overline{\nabla}_{V_1} \overline{R})(V_2, X, V_3, V_4) + (\overline{\nabla}_{V_2} \overline{R})(X, V_1, V_3, V_4) + \\ & \overline{R}(T_X V_1, V_2, V_3, V_4) + \overline{R}(T_{V_1} V_2, X, V_3, V_4) + \overline{R}(T_{V_2} X, V_1, V_3, V_4) = 0. \end{aligned}$$

Now, the second and the third terms of this equation are vanishing by lemma 5.1, (iii). It is easy to see that the first term is J -invariant in V_1 and V_2 and that all the remaining terms are J -anti-invariant in V_1 and V_2 . Therefore, (ii) is proven and we obtain : $\overline{R}(T_X V_1, V_2, V_3, V_4) + \overline{R}(T_{V_1} V_2, X, V_3, V_4) + \overline{R}(T_{V_2} X, V_1, V_3, V_4) = 0$. Since $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$ it suffices now to use lemma 5.1, (ii) to conclude. ■

An important consequence of the equation (5.1) is :

Corollary 5.1 *We have*

$$\eta_{\mathcal{V}_1} \hat{\eta}_{\mathcal{V}_0} \mathcal{V}_0 = 0.$$

Proof :

Take V_3 in \mathcal{V}_0 , V_4 in \mathcal{V}_1 and V_1, V_2 in \mathcal{V} in equation (5.1). Since $\mathcal{V}_i, i = 0, 1$ are orthogonal and $\overline{\nabla}$ -parallel inside \mathcal{V} we have that

$$\langle [\eta_{V_3}, \eta_{V_4}]X, T(V_1, V_2) \rangle = 0.$$

for all X in H . Since by definition $T(\mathcal{V}, \mathcal{V}) = \mathcal{V}_0$ it follows that $\langle [\eta_{V_3}, \eta_{V_4}]X, U \rangle = 0$ for all U in \mathcal{V}_0 . Now, $\eta_{V_4}(\eta_{V_3}X)$ is in H since V_4 is in \mathcal{V}_1 and $\eta_{V_3}X$ in \mathcal{V} hence $\langle \eta_{V_4}X, \hat{\eta}_{V_3}U \rangle = - \langle \eta_{V_3}\eta_{V_4}X, U \rangle = 0$. In our notations, $\langle \eta_{\mathcal{V}_1}H, \hat{\eta}_{\mathcal{V}_0}\mathcal{V}_0 \rangle = 0$. Then we get that $\eta_{\mathcal{V}_1}\hat{\eta}_{\mathcal{V}_0}\mathcal{V}_0$ is orthogonal to H . Or the definition of \mathcal{V}_1 ensures that $\eta_{\mathcal{V}_1}\mathcal{V}$ is contained in H and our result follows. ■

Using the previous corollary and equation (5.1) as main tools, we will proceed now to the refinement of the decomposition given in proposition 4.1. More precisely, our immediate objective will be to show that the space W_2 occurring in the decomposition given in proposition 4.1, must vanish. To proceed, consider the decomposition

$$\mathcal{V}_0 = W_1 \oplus W_2$$

and define $E' = \hat{\eta}_{W_2}W_2 \subseteq \mathcal{V}_1$ and let E be the orthogonal complement of E' in \mathcal{V}_1 . Obviously, the decomposition $\mathcal{V}_1 = E \oplus E'$ is J -invariant and $\overline{\nabla}$ -parallel inside \mathcal{V} .

Using corollary 5.1 we obtain some preliminary algebraic information as follows.

Lemma 5.2 *We have :*

$$\eta_{E'}H \subseteq W_2.$$

Proof :

Using the corollary 5.1, the fact that $T(W_1, W_1) = W_1$ and the definition of E' we obtain easily that

$$\mathbf{5.2} \quad \eta_{\mathcal{V}_1}W_1 = 0$$

and

$$\mathbf{5.3} \quad \eta_{\mathcal{V}_1}E' = 0.$$

The second equation gives us $\eta_E E' = \eta_{E'} E' = 0$. Then we have that $\eta_{E'}E$ is contained in $T(E, E') = 0$ (since $T(\mathcal{V}_1, \mathcal{V}_1) = 0$) and thus it vanishes, showing that $\eta_{E'}\mathcal{V}_1 = 0$. Hence $\eta_{E'}H$ is orthogonal to \mathcal{V}_1 and by the first equation it is also orthogonal to W_1 and our result follows. ■

We will need now one more auxiliary lemma.

Lemma 5.3 *We have :*

(i)

$$\overline{R}(v_1, w_1, v_0, w_0) = - \langle [\hat{\eta}_{v_0}, \hat{\eta}_{w_0}]v_1, w_1 \rangle$$

for all v_1, w_1 in \mathcal{V}_1 and v_0, w_0 in \mathcal{V}_0 .

(ii) Define a symmetric tensor $\hat{r} : \mathcal{V}_0 \rightarrow \mathcal{V}_0$ by setting

$$\langle \hat{r}v_0, w_0 \rangle = \sum_{v_k \in \mathcal{V}_1} \langle \hat{\eta}_{v_0}v_k, \hat{\eta}_{w_0}v_k \rangle$$

for all v_0, w_0 in \mathcal{V}_0 (here $\{v_k\}$ is an arbitrary local orthonormal basis in \mathcal{V}_1). Then \hat{r} preserves W_2 and the restriction of \hat{r} to W_2 has no kernel.

(iii) Let U be in \mathcal{V} such that

$$\overline{R}(U, V_1, V_2, V_3) = 0$$

for all V_1 in E' and V_2, V_3 in \mathcal{V}_1 . Then $U \perp E'$.

Proof :

(i) It suffices to apply lemma 4.4, (iii) and (iv).

(ii) That W_2 is preserved by \hat{r} follows directly by the fact that W_2 is $\overline{\nabla}$ -parallel inside \mathcal{V} . Let v in W_2 be such $\hat{r}v = 0$. Then the definition of \hat{r} implies directly that $\hat{\eta}_v \mathcal{V}_1 = 0$. It follows that $\hat{\eta}_v W_2$ is orthogonal to \mathcal{V}_1 and we know that it is also contained in \mathcal{V}^1 , since v belongs to W_2 (cf. proposition 4.1, (iii)). Thus $\hat{\eta}_v W_2 = 0$, and again the fact that v belongs to W_2 yields $\hat{\eta}_v W_1 = 0$ (see proposition 4.1, (ii)). Hence $\hat{\eta}_v \mathcal{V} = 0$, v is in $W_2 \subseteq \mathcal{V}_0$ and this clearly implies the vanishing of v .

(iii) Using the symmetry formula of lemma 4.4, (iv) we obtain that

$$\overline{R}(V_2, V_3, V_1, U) = 0$$

for all V_2, V_3 in \mathcal{V}_1 and V_1 in E' . Let now v_0, w_0 be in W_2 . If $\{v_k\}$ is an orthonormal basis in \mathcal{V}_1 then by lemma 4.2, (i) we have that :

$$5.4 \quad \overline{R}(v_k, Jv_k)\eta_{v_0}w_0 = \eta_{\overline{R}(v_k, Jv_k)v_0}w_0 + \eta_{v_0}(\overline{R}(v_k, Jv_k)w_0).$$

Now note that using point (i), one easily finds that $\sum_{v_k \in \mathcal{V}_1} \overline{R}(v_k, Jv_k)v = -2(\hat{r}J)v$ for all v in \mathcal{V}_0 . With this in mind, we project (5.4) on \mathcal{V} and sum over k to find :

$$\sum_{v_k \in \mathcal{V}_1} \overline{R}(v_k, Jv_k)\hat{\eta}_{v_0}w_0 = 2J[\hat{\eta}_{\hat{r}v_0}w_0 + \hat{\eta}_{v_0}(\hat{r}w_0)].$$

But elements of the form $\hat{\eta}_{v_0}w_0$ generates, by definition, E' hence taking the scalar product with U yields to $\langle \hat{\eta}_{\hat{r}v_0}w_0 + \hat{\eta}_{v_0}(\hat{r}w_0), JU \rangle = 0$ for all v_0, w_0 elements of W_2 . Since \hat{r} is positive without kernel on W_2 , by eventually considering its spectral decomposition we deduce that JU (and then U) is orthogonal to $\hat{\eta}_{W_2}W_2 = E'$ and the proof is finished. ■

Based on these preparations we are able now to show the following.

Proposition 5.2 *The subbundle \mathcal{V} admits a orthogonal, J -invariant and $\overline{\nabla}$ -parallel (inside \mathcal{V})-decomposition :*

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$$

with $\mathcal{V}_0 = T(\mathcal{V}_0, \mathcal{V}_0)$ and $\eta_{\mathcal{V}_1} \mathcal{V}_0 = 0$.

Proof :

We are going to show first that we must have $E' = 0$. Let us consider v_0 in W_2 , w_0 in E' and v_1, v_2 in \mathcal{V}_1 as well as X in H . Using proposition 5.1, (i) we obtain

$$\overline{R}(\eta_{v_0} X, w_0, v_1, v_2) - \overline{R}(\eta_{w_0} X, v_0, v_1, v_2) = - \langle [\eta_{v_1}, \eta_{v_2}] X, T_{w_0} v_0 \rangle .$$

But $\eta_{w_0} X$ belongs (see lemma 5.2) to $W_2 \subseteq \mathcal{V}_0$, hence the second term of the left hand side vanishes by lemma 4.4, (iii), as well as the right hand side, since $T_{w_0} v_0$ is in \mathcal{V} whilst $[\eta_{v_1}, \eta_{v_2}] X$ is in H . We found that :

$$\overline{R}(\eta_{v_0} X, w_0, v_1, v_2) = 0$$

for all w_0 in E' and all v_1, v_2 in \mathcal{V}_1 . Applying lemma 5.3, (iii), we obtain that $\eta_{v_0} H$ is orthogonal to E' , that is in our notations $\langle \eta_{W_2} H, E' \rangle = 0$. It follows that $\eta_{W_2} E' \subseteq \mathcal{V} = W_1 \oplus W_2 \oplus \mathcal{V}_1$. But $\hat{\eta}_{W_2} W_1 = 0, \hat{\eta}_{W_2} \mathcal{V}_1 = W_2$ (cf. proposition 4.1, (ii) and (iv)) hence $\eta_{W_2} E' \subseteq W_2$. Or $T(E', W_2) \subseteq \mathcal{V}$ hence it follows that $\eta_{E'} W_2 \subseteq \mathcal{V}$. Since E' lies in \mathcal{V}_1 , the definition of \mathcal{V}_1 implies $\eta_{E'} W_2 \subseteq H$ and we obtain that $\eta_{E'} W_2 = 0$. By means of lemma 5.2 this also implies $\eta_{E'} H = 0$. But using (5.2) and (5.3) we obtain (because E' is contained in \mathcal{V}_1) that $\eta_{E'} W_1 = 0$ and $\eta_{E'} \mathcal{V}_1 = 0$. This means that E' is contained in H , the Kähler nullity of (g, J) and we obtain that $E' = 0$.

Now, $\hat{\eta}_{W_2} W_2$ must vanish and it follows that $\hat{\eta}_{W_2} \mathcal{V}_1$ is orthogonal to W_2 . But we already know (see proposition 4.1, (iv)) that the last mentioned spaces are in fact equal, and then $W_2 = 0$. It follows that $W_1 = \mathcal{V}_0$ and the proof is now finished, where the last assertion follows by (5.2). ■

We will investigate now the geometrical properties of the decomposition in the proposition 5.2, properties that will lead, once again, to a better understanding of its algebraic structure.

Let us introduce the *configuration* tensor $A : H \times H \rightarrow \mathcal{V}$ by setting :

$$\overline{\nabla}_X Y = \tilde{\nabla}_X Y + A_X Y$$

for all X, Y in H , where $\tilde{\nabla}_X Y$ denotes the orthogonal projection of $\overline{\nabla}_X Y$ on H . The tensor A is precisely the obstruction to the distribution H to be $\overline{\nabla}$ -parallel. In a similar way, we define $B : H \times \mathcal{V} \rightarrow H$ by

$$\overline{\nabla}_X V = \tilde{\nabla}_X V + B_X V.$$

Because the connection $\overline{\nabla}$ is metric we have $\langle B_X V, Y \rangle = - \langle V, A_X Y \rangle$ for all X, Y in H and V in \mathcal{V} . Since H is integrable we have that A is symmetric, that is $A_X Y = A_Y X$. It is also obvious that $[A_X, J] = 0$ for all X in H .

Lemma 5.4 (i) *The distribution \mathcal{V}_0 is $\bar{\nabla}$ -parallel.*

(ii) *Moreover we have that $\eta_{\mathcal{V}_0}\mathcal{V}_1 = 0$ and $\eta_{\mathcal{V}_1}H = \mathcal{V}_1$.*

Proof :

(i) We begin by showing that the operator B_X, X in H must vanish on \mathcal{V}_0 . Indeed, let us recall that $(\bar{\nabla}_X\eta)(V, \cdot) = (\bar{\nabla}_V\eta)(X, \cdot) = 0$ for all X in H and V in \mathcal{V} (see (3.6) and (3.13)). If W is in \mathcal{V} it follows then easily that $(\bar{\nabla}_X T)(V, W) = 0$. In other words

$$\tilde{\nabla}(T_V W) + B_X(T_V W) = T(\bar{\nabla}_X V, W) + T(V, \bar{\nabla}_X W)$$

belongs to \mathcal{V} and this implies the vanishing of B_X on \mathcal{V}_0 . Consider now V_0, W_0 in \mathcal{V}_0 . We have, for any X in H :

$$\begin{aligned} \bar{\nabla}_X(T_{V_0}W_0) &= T(\bar{\nabla}_X V_0, W_0) + T(V_0, \bar{\nabla}_X W_0) \\ &= T(\tilde{\nabla}_X V_0, W_0) + T(V_0, \tilde{\nabla}_X W_0). \end{aligned}$$

But $\mathcal{V}_0 = T(\mathcal{V}_0, \mathcal{V}_0) = T(\mathcal{V}_0, \mathcal{V})$ and \mathcal{V}_0 is $\bar{\nabla}$ -parallel insider \mathcal{V} , hence (i) is proven.

(ii) We will show first that $\eta_{\mathcal{V}_1}H = \mathcal{V}_1$. Since $\eta_{\mathcal{V}_1}\mathcal{V}_0 = 0$ we have that $\eta_{\mathcal{V}_1}H \subseteq \mathcal{V}_1$. Consider the decomposition $\mathcal{V}_1 = E \oplus F$ with $\eta_{\mathcal{V}_1}H = E$ and F the orthogonal complement of E in \mathcal{V}_1 . From the definition of F it follows that $\eta_{\mathcal{V}_1}F$ is orthogonal to H and hence it vanishes (recall that $\eta_{\mathcal{V}_1}\mathcal{V}_1 \subseteq H$). Since $T(\mathcal{V}_1, \mathcal{V}_1) = 0$ it also follows that $\eta_F\mathcal{V}_1 = 0$. This implies that $\eta_F H$, which a subspace of \mathcal{V}_1 , is orthogonal to \mathcal{V}_1 , and hence $\eta_F H = 0$. Finally since $\eta_F\mathcal{V}_0 = 0$ (F lies in \mathcal{V}_1) we get that F is contained in the Kähler nullity of (g, J) and then, of course, $F=0$.

Now, by (i) we deduce that $\mathcal{V}_1 \oplus H$ is a $\bar{\nabla}$ -parallel distribution. Using an argument similar to that of lemma 4.4, (ii) for the $\bar{\nabla}$ -parallel decomposition $TM = \mathcal{V}_0 \oplus (\mathcal{V}_1 \oplus H)$ we find that

$$\eta_{\mathcal{V}_0}T(\mathcal{V}_1, H) = \eta_{\mathcal{V}_0}\eta_{\mathcal{V}_1}H = 0.$$

Combining this with $\eta_{\mathcal{V}_1}H = \mathcal{V}_1$ finishes the proof of the lemma. ■

The last step before proving the splitting theorem 1.1, consists in investigating reducibility properties of the Kähler nullity H . We need to introduce some notations. For every X in H define a linear map :

$$\gamma_X : \mathcal{V}_1 \rightarrow \mathcal{V}_1 \text{ by } \gamma_X V = \eta_V X.$$

The maps γ_X are in relation with the curvature of H (with respect to the canonical Hermitian connection), as showed in the following lemma.

Lemma 5.5 *Let X, Y, Z be in H and V, W in \mathcal{V}_1 . We have :*

$$\bar{R}(X, Y, \eta_V W, Z) = \bar{R}(\gamma_Z V, W, X, Y) + \langle [[\gamma_X, \gamma_Y], \gamma_Z]V, W \rangle$$

Proof :

From lemma 4.2, (ii), we know that $[\bar{R}(X, Y), \eta_V]W = \eta_{\beta_V(X, Y)}W$, where, in our present notations $\beta_V(X, Y) = \eta_{\gamma_Y}VX - \eta_{\gamma_X}VY = [\gamma_X, \gamma_Y]V$. Taking the scalar product with Z yields after an easy calculation to the wanted result. ■

Lemma 5.6 *Suppose that H_1 is a J -invariant, $\overline{\nabla}$ -parallel distribution contained in H . Let H_2 be the orthogonal complement of H_1 in H . Then the spaces $\eta_{\mathcal{V}_1}H_1$ and $\eta_{\mathcal{V}_1}H_2$ are mutually orthogonal.*

Proof :

Let X_1 and X_2 be in H_1 and H_2 respectively. Then the parallelism of H_1 , together with the symmetry property of \overline{R} (see corollary 3.1, (i)) ensures that $\overline{R}(X_1, X_2, \eta_{\mathcal{V}_1}W, Z) = \overline{R}(\gamma_Z V, W, X_1, X_2) = 0$ for all V, W in \mathcal{V}_1 and Z in H . Then, by the previous lemma we obtain

$$[[\gamma_{X_1}, \gamma_{X_2}], \gamma_Z] = 0$$

for all Z in H . Taking $Z = X_1$ we find that

$$\gamma_{X_1}^2 \gamma_{X_2} + \gamma_{X_2} \gamma_{X_1}^2 = 2\gamma_{X_1} \gamma_{X_2} \gamma_{X_1}.$$

We change now X_2 in JX_2 in the previous equation and take into account that $\gamma_{JX} = \gamma_X J = -J\gamma_X$. It follows that

$$\gamma_{X_1}^2 \gamma_{X_2} + \gamma_{X_2} \gamma_{X_1}^2 = -2\gamma_{X_1} \gamma_{X_2} \gamma_{X_1}$$

hence we must have

$$\gamma_{X_1}^2 \gamma_{X_2} + \gamma_{X_2} \gamma_{X_1}^2 = \gamma_{X_1} \gamma_{X_2} \gamma_{X_1} = 0.$$

This easily implies that $\gamma_{X_1}^3 \gamma_{X_2} = 0$ and since γ_X is a symmetric operator for all X in H (as a consequence of $T(\mathcal{V}, \mathcal{V}) \subseteq \mathcal{V}$) we get that $\gamma_{X_1} \gamma_{X_2} = 0$. But this fact is equivalent to the orthogonality of the spaces $\eta_{\mathcal{V}_1}H_1$ and $\eta_{\mathcal{V}_1}H_2$, and the proof is now finished. ■

Proof of theorem 1.1 :

Let us define the distribution H_1 to be $H_1 = (\eta_{\mathcal{V}_0} \mathcal{V}_0)_H$ where the subscript denotes orthogonal projection. We have then $\eta_{\mathcal{V}_0} \mathcal{V}_0 = \mathcal{V}_0 \oplus H_1$. As \mathcal{V}_0 is $\overline{\nabla}$ -parallel, so is $\eta_{\mathcal{V}_0} \mathcal{V}_0$, hence H_1 must be $\overline{\nabla}$ -parallel. Define now $W_i = \eta_{\mathcal{V}_1} H_i, 1 \leq i \leq 2$. Then using lemma 5.6 and the fact that $\eta_{\mathcal{V}_1} H = \mathcal{V}_1$ we obtain a J -invariant and orthogonal decomposition

$$\mathcal{V}_1 = W_1 \oplus W_2.$$

The orthogonality of W_1 and W_2 ensures, in the standard way, that $\eta_{W_1} W_2 = 0$ and that $\eta_{W_1} H_2 = \eta_{W_2} H_2 = 0$. Let us show now that W_1 is a $\overline{\nabla}$ -parallel distribution. Let X be in H and U, X_1 be in W_1 and H_1 respectively. Then :

$$\overline{\nabla}_X (\eta_U X_1) = \eta_U \overline{\nabla}_X X_1 + \eta_{\overline{\nabla}_X U} X_1$$

belongs to $\eta_{W_1} H_1 + \eta_{TM} X_1 = W_1$, were we used the $\overline{\nabla}$ -parallelism of H_1 . In the same way it can be showed that W_1 is $\overline{\nabla}$ -parallel inside \mathcal{V} . We get a $\overline{\nabla}$ -parallel decomposition :

$$TM = (\mathcal{V}_0 \oplus W_1 \oplus H_1) \oplus (W_2 \oplus H_2).$$

Using the behaviour of the tensor η with respect to this decomposition proven previously it follows that this splitting is in fact ∇ -parallel. From the discussion below it is now clear that the first factors gives rise to a \mathcal{AK} -manifold with parallel torsion and the second to a special \mathcal{AK}_2 -manifold. ■

6 On parallel torsion

In this section we will consider an almost Kähler manifold (M^{2n}, g, J) whose first canonical connection has parallel torsion. Then (M^{2n}, g, J) naturally belongs to the class \mathcal{AK}_2 . Our aim here is to show that in this setting that metric cannot be Einstein if the J is not integrable. Along the way we will also obtain some information about the holonomy of the canonical Hermitian connection. We will need first some preliminaries.

Lemma 6.1 *For any \mathcal{AK} -manifold with parallel torsion we have :*

(i)

$$\nabla^* \nabla \omega = \Phi^0$$

where the 2-form Φ^0 is defined by $\Phi^0(X, Y) = \langle (rJ)X, Y \rangle$.

(ii) If $\{e_i, 1 \leq i \leq 2n\}$ is a local orthonormal basis in TM , then :

$$\bar{\rho} = \sum_{i=1}^{2n} \bar{R}(e_i, Je_i) = 2\rho + \frac{1}{2}\Phi^0.$$

Proof :

(i) Let us recall the fact that $(\nabla_X \omega)(Y, Z) = \langle (\nabla_X J)Y, Z \rangle$ for all X, Y, Z in TM . Using this and the parallelism of the torsion, a simple algebraic computation which will left to the reader yields to the desired result.

(ii) Follows immediately from formula (3.11). ■

It is an easy exercise to show that the Ricci tensor is J -invariant in the presence of parallel torsion. Furthermore, from lemma 6.1, (ii) it follows also that the Ricci- \star tensor is J -invariant.

Now, our key ingredient for proving theorem 1.2 consists in the following lemma, whose first part is Sekigawa's formula in the \mathcal{AK}_2 case and whose second part is a relation complementary to Sekigawa's coming from the parallelism of the torsion.

Lemma 6.2 *We have :*

(i)

$$4 \langle \rho, \Phi - \Phi^0 \rangle = |\Phi|^2 + |\Phi^0|^2$$

(ii)

$$4 \langle \rho, \Phi + \Phi^0 \rangle + \langle \Phi^0, \Phi + \Phi^0 \rangle = 0.$$

Proof :

(i) This is an immediate consequence of Sekigawa's formula (see proposition 2.1), actualized in the parallel torsion case. Indeed, we must have $\bar{R}' = 0$ (see corollary 3.2). It also clear that $s^* - s$ is a constant function. Then, using lemma 6.1, (i) in Sekigawa's formula we get the desired result.

(ii) Since the torsion is parallel, we have that $\bar{R}(X, Y) \cdot \eta = 0$ for all X, Y in TM . It follows that $\bar{\rho} \cdot \eta = 0$ and taking the scalar product with $J\eta$ we obtain after an easy computation that $\langle \bar{\rho}, \Phi + \Phi^0 \rangle = 0$. It suffices now to use lemma 6.1, (ii) to conclude. ■

par We can prove the following proposition, which is nothing else than theorem 1.2 in the introduction.

Proposition 6.1 *Let (M^{2n}, g, J) be almost Kähler with parallel torsion. Then :*

- (i) *If g is Einstein then J is integrable ;*
- (ii) *If J is not integrable the connection $\bar{\nabla}$ has real reducible holonomy.*

Proof :

We prove both assertions in the same time. Let us suppose that we have

$$6.1 \quad 2 \langle \rho, \Phi \rangle = \langle \rho, \Phi^0 \rangle$$

and prove that J is integrable. Using (7.1), the relations in lemma 7.1 become :

$$\begin{aligned} -2 \langle \rho, \Phi^0 \rangle &= |\Phi|^2 + |\Phi^0|^2 \\ 6 \langle \rho, \Phi^0 \rangle &+ |\Phi^0|^2 + \langle \Phi^0, \Phi \rangle = 0. \end{aligned}$$

We deduce that $3|\Phi|^2 + 2|\Phi^0|^2 = \langle \Phi, \Phi^0 \rangle$. Since $\langle \Phi, \Phi^0 \rangle \leq |\Phi| \cdot |\Phi^0|$ we have clearly that $\Phi^0 = \Phi = 0$, that is (g, J) is a Kähler structure.

Now, if the manifold is Einstein, (7.1) is clearly satisfied, hence (i) is proven. To prove (ii), suppose that $\bar{\nabla}$ has irreducible holonomy. Then the $\bar{\nabla}$ -parallel forms Φ, Φ^0 must be multiples of ω hence $2\Phi = \Phi^0$ so (7.1) is again satisfied. ■

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