Reduced 1-cohomology of connected locally compact groups and applications

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Abstract

In this article we will focus on the reduced-1 cohomology spaces of locally compact connected groups with coefficients in unitary representations. The vanishing of these spaces for every unitary irreducible representation characterizes the Kazhdan's property (T). The main theorem state that for a connected locally compact group, there are only a finite number of unitary irreducible representation for which the reduced 1-cohomology does not vanish. Moreover, a description of these representations is given.

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1 Introduction

The vanishing of the reduced-1 cohomology spaces for every unitary irreducible representation characterizes the Kazhdan's property (T) (see Y.Shalom [15]) for a compactly generated group (in particular a connected group). For the connected solvable Lie groups, P. Delorme established the following theorem ([4]):

Theorem 1.1. *(Delorme) For every irreducible representations of a connected solvable Lie group of degree at least 2, the reduced-*1 *cohomology vanishes. Moreover there are only finitely many characters for which the reduced-*1 *cohomology is not zero.*

As (non compact) solvable Lie groups do not have property (T) we can interpret this result by saying that the lack of property (T) of such groups is, from a cohomological point of view, concentrated in the 1-dimensional representations.
The main goal of this paper is to understand for connected Lie groups where the lack of property (T) is concentrated. Delorme's theorem provides the answer for connected solvable Lie groups. This is done by enlarging the class of solvable groups in several steps

First, in section 3, we treat the class of amenable groups. We show that if G is an amenable connected locally compact group then the only irreducible representations π of G which carry reduced 1-cohomology are finite dimensional and there are only finitely many such representations. It shows in particular that such a group has the property (H_{FD}) (defined by Y.Shalom in [14]).

Theorem 3.4 *Let G a locally compact almost connected amenable group. The unitary irreducible representations with non vanishing reduced* 1*-cohomology are all finite dimensional and there are only finitely many such representations.*

A nice corollary of this fact is the vanishing of $\overline{H^1}(G, L^2(G))$ for these groups. As this vanishing result is also true for discrete amenable groups (see [12]), we conjecture that $\overline{H^1}(G, L^2(G))$ is zero for every amenable locally compact group.

In section 4, we are interested in a much larger class of groups, namely the groups having the Haagerup property. Recall that a locally compact group G has the Haagerup property if there exists a proper conditionally negative definite function on G. For such connected locally compact groups, we show that there are only finitely many irreducible representations which characterize the lack of property (T). However these representations are not finite dimensional in general.
Finally we give a description of the irreducible representations of a locally compact connected group for which the associated 1-cohomology space is not trivial:

Theroem 6.4 *Let G be a almost connected locally compact group. Then there are only finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G, \pi)$.*
Moreover, if G does not have property (T) (which implies the existence of an irreducible unitary representation π of G with $\overline{H^1}(G, \pi) \neq 0$), any such non trivial representation π factors through an irreducible unitary representation σ of a group H isomorphic to $PO(n, 1), PU(m, 1)$ or to a non-compact amenable non-nilpotent group H such that $\overline{H^1}(H, \sigma) \cong \overline{H^1}(G, \pi) \neq 0$.

The study of irreducible representations π of a group G for which $H^1(G, \pi) \neq 0$ is motivated by the Vershik-Karpushev theorem (see [17] and [11]). Let us recall that the support of a representation π of a group G is the set of irreducible representations of G which are weakly contained in π and that the cortex of the group, $\mathrm{Cor}\,(G)$, is the set of all irreducible representation which are not separated from the trivial representation for the Fell-Jacobson topology on the dual space $\widehat{G}$ (see [11] for a nice presentation of this). The Vershik-Karpushev theorem is:

Theorem 1.2. *If π is a unitary factorial representation of a second countable locally compact group G with $H^1(G, \pi) \neq 0$ then $supp\, \pi \subset Cor(G)$.*

We can interpret this result by saying that the lack of property (T) is topologically concentrated in the cortex of the group. Here we are rather interested in a more algebraic characterization of these representations, but for reduced cohomology.

Another motivation is given by Guichardet's property (P) (see [6]):

A locally compact group G has property (P) if the set of irreducible unitary representations π for which $\overline{H^1}(G,\pi) \neq 0$ is finite and all its elements are closed points in $\widehat{G}$.

In the original definition of this property, we also want these representations to be non-separated from the trivial representation. As this condition is a direct consequence of the Vershik-Karpushev's theorem we omitted it from the definition.

In the last section, we apply these vanishing results to the study of smooth μ-harmonic Dirichlet finite functions on smooth manifold which are homogeneous spaces of connected unimodular Lie groups. We show that if G is a unimodular connected Lie group acting transitively on a smooth connected non-compact manifold M with $\overline{H^1}(G, L^2(M)) = 0$, and if μ is a symmetric probability measure on G whose support is a compact generating set of G, then the only smooth Dirichlet-finite μ-harmonic functions on M are the constant functions. In [12], the authors proved the analogous result in the case where the groups are discrete. In [1], G.Alexopoulos proved this kind of result for the bounded functions on discrete polycyclic groups.

2 1-cohomology and reduced-1 cohomology

Let G be a locally compact σ-compact separable group and let $(\pi, \mathcal{H}_\pi)$ be a strongly continuous unitary representation of G.

Definition 2.1.

1) *A continuous map $b : G \to \mathcal{H}_\pi$ is a 1-cocycle with respect to π if it satisfies the following relation:*

$$b(gh) = b(g) + \pi(g)b(h) \ (*)$$

for all $g, h \in G$.
The space of cocycles endowed with the topology of uniform convergence on compact sets of G is a Fréchet space, denoted by $Z^1(G,\pi)$.

2) *A cocycle b is a coboundary if there exists an element $\xi \in \mathcal{H}_\pi$ such that $b(g) = \pi(g)\xi - \xi$. The set of coboudaries is a subspace of $Z^1(G,\pi)$*

denoted by $B^1(G,\pi)$. The closure of the coboundaries in $Z^1(G,\pi)$ will be denoted by $\overline{B^1}(G,\pi)$. An element of this space is called an almost coboundary.

3) *The* 1*-cohomology of G with coefficients in π is the quotient space*

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

4) *The* 1- reduced-*cohomology of G with coefficient in π is the Hausdorff quotient space*

$$\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1}(G,\pi).$$

We have a nice geometrical interpretation of these spaces in terms of affine isometric actions of the group G.

Definition 2.2. *Let $\mathcal{H}$ be an affine Hilbert space. An affine isometric action of G on $\mathcal{H}$ is a strongly continuous group homomorphism $\alpha : G \to Is(\mathcal{H})$ to the group of affine isometries of $\mathcal{H}$.*

The next lemma establishes a relationship between affine isometric actions, unitary representations and 1-cocycles.

Lemma 2.3. *Any affine isometric action $\alpha : G \to Is(\mathcal{H})$ can be written as $\alpha(g)v = \pi(g)v + b(g)$ $(v \in \mathcal{H})$ where π is an unitary representation of G on the underlying Hilbert space of $\mathcal{H}$ and $b : G \to \mathcal{H}$ is a* 1*-cocycle.The representation π is called the linear part of α and b is the translation part of α. Conversely, given π an unitary representation of G on a Hilbert space $\mathcal{H}$ and $b : G \to \mathcal{H}$ a* 1*-cocycle, we can define an affine isometric action by setting $\alpha(g)\xi = \pi(g)\xi + b(g)$, $\forall \xi \in \mathcal{H}_\pi$.*

For the proof, we refer to [10]. It is an easy exercise to show that, given an unitary representation π of G, the coboundaries $b \in B^1(G,\pi)$ correspond to affine isometric actions with linear part π which have fixed points. Moreover, almost coboundaries $b \in \overline{B^1}(G,\pi)$ correspond to those actions α which almost have fixed points in the sense that for every $\varepsilon > 0$ and for every compact subset K of G, there exists an element $\xi \in \mathcal{H}_\pi$ such that

$$\max_{k \in K} \|\alpha(k)\xi - \xi\| < \varepsilon$$

Hence the following interpretations:

- The 1-cohomology space $H^1(G,\pi)$ classifies the affine isometric actions of G with linear part π which have a fixed point.

- The reduced-1 cohomology space $\overline{H^1}(G,\pi)$ classifies the affine isometric actions of G with linear part π which almost have fixed points.

2.1 Some properties

For a given unitary representation π of a locally compact group G, one can ask if the associated 1-cohomology and reduced-1 cohomology coincide. The answer is given by A.Guichardet in [5]:

Proposition 2.4. *Let π be a unitary representation of G without non zero invariant vectors. The following are equivalent:*

i) π does not almost have invariant vectors (i.e. there exists $\varepsilon > 0$,a compact subset K of G, such that $\max\limits_{k\in K}\|\pi(k)\xi - \xi\| \geq \varepsilon\|\xi\|$) for all $\xi \in \mathcal{H}_\pi$;

ii) $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$;

iii) $H^1(G,\pi) = \overline{H^1}(G,\pi)$.

In the case where π has a non zero invariant vector, one can decompose it as an orthogonal direct sum of the form $\pi_0 \oplus 1$ where π_0 doesn't have non zero invariant vectors and where 1 denote the trivial action of G on $\mathcal{H}_\pi$. As $H^1(G,1) = \overline{H^1}(G,1) = Z^1(G,1)$, one can compare the 1-cohomology with the reduced-1 cohomology spaces by using the following property (see for example [7]):

Lemma 2.5. *Let $\pi_1, ..., \pi_n$ be a finite set of unitary representations of a group G. Then*

$$H^1(G, \oplus_{i=1}^n \pi_i) = \oplus_{i=1}^n H^1(G,\pi_i)$$

Remark that this statement is no longer true in general for an infinite family of unitary representations. However, we have:

Lemma 2.6. *([2]) If π is a unitary representation of a locally compact group G, then*

$$H^1(G,\pi) = 0 \Leftrightarrow H^1(G, \infty\cdot\pi) = 0$$

If we deal with reduced-1 cohomology these kind of properties behave quite nicely (see [5]):

Proposition 2.7. *Let (X,μ) be a measured space and $(\pi_x)_{x\in X}$ a measurable field of unitary representations of a locally compact group G. If $\overline{H^1}(G,\pi_x)=0$ for μ-almost every $x\in X$, then*

$$\overline{H^1}(G,\int_X^{\oplus}\pi_x\,d\mu(x))=0.$$

2.2 Normal subgroups

The aim of this section is to study rigidity phenomenon of the following type: Let α be an affine isometric action of a locally compact group G and N a closed normal subgroup of G. If the restriction of the action to N admits a fixed point (resp. almost fixed points) what can be said about the existence of a G-fixed point (resp. almost fixed points for α)?
How does the behaviour of an affine isometric action on a normal subgroup influence the global behaviour of the action? What is the link between affine isometric actions of the group G and those of the normal subgroup N?

Lemma 2.8. *Let N be a closed normal subgroup of a locally compact group G and α an affine isometric action of G whose linear part doesn't have non zero N-invariant vectors. If the restriction of α to N has a fixed point, then α has a fixed point.*

We can give a short geometrical proof if this fact:
Let α be an affine isometric action with linear part π,whose restriction to N has a fixed point. Let $\mathcal{H}^N$ be the set of $\alpha(N)$-fixed points. If $\xi,\eta\in\mathcal{H}^N$, then $\xi-\eta=\alpha(n)\xi-\alpha(n)\eta=\pi(n)(\xi-\eta)$. But we assume π not to have N-invariant non zero vectors; so we conclude that $\mathcal{H}^N$ is reduced to a single point. On the other hand, $\mathcal{H}^N$ is $\alpha(G)$-invariant by normality of N in G.

The preceding lemma can be also stated as : *Let N be a closed normal subgroup of G and π a unitary representation of G without non zero N-invariant vectors. Then the restriction map induced by restriction of cocycles from G to N, Res : $H^1(G,\pi)\to H^1(N,\pi)$ is injective.*

The analogous statement of lemma 2.8 in the context of non-reduced cohomology is not true in general (see [12]). Under cocompactness condition on the normal subgroup, we can state:

Proposition 2.9. *Let G be a locally compact group and N a closed, normal, cocompact subgroup of G. Let π be a unitary representation of G. Then the restriction map $Res : \overline{H^1}(G,\pi) \to \overline{H^1}(N,\pi|_N)$ is injective.*
In particular if $\overline{H^1}(N,\pi|_N) = 0$ then $\overline{H^1}(G,\pi) = 0$.

Proof. By [9], there exists a Borel regular section $s : G/N \to G$ whose image is relatively compact. For all $x \in G/N$, and all $g \in G$, $gs(x)$, $s(gx)$ has the same image in G/N. Let us define a cocycle ("à la Zimmer") $\sigma : G/N \times G \to N$; $\sigma(x,g) = (s(gx)^{-1}gs(x))^{-1}$. So that $\sigma(x,g)$ is the unique element of N satisfying $gs(x)\sigma(x,g) \in s(G/N)$. Remark that $\sigma(G/N,K)$ is relatively compact whenever K is a compact subset of G.
Let α be an affine action of G such that $\alpha|_N$ almost has fixed points and let us show that it almost has fixed points.
Let K be a compact subset of G, it is contained in a compact subset of the form $K_0\ \overline{s(G/N)}$, where K_0 is a compact subset of N. Let K_X be the compact subset (by normality of N) of N defined by:

$$K_X = \mathrm{Adh}_N\{s(x)^{-1}ns(x)\sigma(x,x_0^{-1}) \,|\, n \in K_0,\ x \in G/N,\ x_0 \in s(G/N)\}.$$

Then for $\varepsilon > 0$ fixed, there exists by assumption a point ξ such that

$$\sup_{n \in K_X} \parallel \alpha(n)\xi - \xi \parallel < \varepsilon.$$

Denote by dx the finite G-invariant normalized measure (for the action $g \cdot s(x) = gs(x)\sigma(x,g)$) induced by the Haar measure on G/N.
For $g_0 \in G$, there exists a unique $x_0 \in s(G/N)$ and a unique $n_0 \in N$ such that $g_0 = n_0x_0$. For $g_0 \in K$, we have:

$$\begin{aligned}
& \parallel \alpha(g_0)\int_{G/N}\alpha(s(x))\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \\
= & \parallel \alpha(n_0x_0)\int_{G/N}\alpha(s(x))\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \\
= & \parallel \alpha(n_0)\int_{G/N}\alpha(x_0s(x)\sigma(x,x_0)\sigma(x,x_0)^{-1})\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \\
= & \parallel \alpha(n_0)\int_{G/N}\alpha(x_0 \cdot s(x)\sigma(x,x_0)^{-1})\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel
\end{aligned}$$

$$
\begin{aligned}
&= \parallel \alpha(n_0)\int_{G/N}\alpha(s(x)\sigma(x_0^{-1}\cdot x, x_0)^{-1})\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \\
&= \parallel \alpha(n_0)\int_{G/N}\alpha(s(x)\sigma(x, x_0^{-1}))\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \\
&= \parallel \int_{G/N}\alpha(n_0 s(x)\sigma(x, x_0^{-1}))\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \\
&= \parallel \int_{G/N}\alpha(n_0 s(x)\sigma(x, x_0^{-1}))\xi - \alpha(s(x))\xi dx \parallel \\
&\leq \sup_{x\in G/N} \parallel \alpha(n_0 s(x)\sigma(x, x_0^{-1}))\xi - \alpha(s(x))\xi \parallel \\
&= \sup_{x\in G/N} \parallel \alpha(s(x)^{-1} n_0 s(x)\sigma(x, x_0^{-1}))\xi - \xi \parallel .
\end{aligned}
$$

So,

$$
\sup_{g_0\in K} \parallel \alpha(g_0)\int_{G/N}\alpha(s(x))\xi dx - \int_{G/N}\alpha(s(x))\xi dx \parallel \leq \sup_{n\in K_X} \parallel \alpha(n)\xi - \xi \parallel < \varepsilon.
$$

■

Corollary 2.10. *Let G and N be as in the previous proposition. Then for every unitary representation π of N:*

$$
\overline{H^1}(N,\pi) = 0 \Rightarrow \overline{H^1}(G, Ind_N^G\pi) = 0.
$$

Proof. This follows from the weel known fact that $(\mathrm{Ind}_N^G\pi)|_N = [G:H]\,\pi$. So if $\overline{H^1}(N,\pi) = 0$, then by the proposition, $\overline{H^1}(N, (\mathrm{Ind}_N^G\pi)|_N) = 0$ and we conclude by proposition 2.9. ■

The remaining part of this section is devoted to recalling some results of Guichardet ([5]) which describe the relationship between the 1-cohomology (resp. reduced) of a group G and the 1-cohomology (resp. reduced 1-cohomology) of a quotient by a closed normal subgroup, with value in a unitary representation of G which is trivial on this normal subgroup .

Theorem 2.11. *Let G be a locally compact group, N a closed normal subgroup of G and π a unitary representation of G such that $\pi|_N = 1$. Then:*

i) Let $A(G, N, \pi)$ be the image of the restriction map from $Z^1(G, \pi)$ to $Z^1(N, 1)$, we have the isomorphisms:

$$H^1(G, \pi) \cong H^1(G/N, \dot{\pi}) \oplus A(G, N, \pi);$$

$$\overline{H^1}(G, \pi) \cong \overline{H^1}(G/N, \dot{\pi}) \oplus A(G, N, \pi).$$

notice that $A(G, N, \pi)$ is contained in $Hom_G(N, \pi)$, the space of G-equivariant homomorphisms from N to the additive group $\mathcal{H}_\pi$.

ii) If G is the semi-direct product $N \rtimes H$ for some H, then $A(G, N, \pi) = Hom_G(N, \pi)$.

As an immediate corollary, we have:

Corollary 2.12. *Let K be a compact normal subgroup of a locally compact group G and let π a unitary representation of G which is trivial on K. Then:*

$$H^1(G, \pi) \cong H^1(G/K, \pi)$$

and

$$\overline{H^1}(G, \pi) \cong \overline{H^1}(G/K, \pi).$$

Let K be the closed normal subgroup of $N/\overline{[N, N]}$ generated by the closure of the union of the compact subgroups, and set $V = (N/\overline{[N, N]})/K$. The group G acts by conjugation on N and this give rise to an action of G on V. The latter factors through an action of G/N on V which will be denoted by σ. Every continuous morphism f from N to $\mathcal{H}_\pi$ factor through a continuous morphism $\tilde{f}$ from V to $\mathcal{H}_\pi$, and f belongs to $Hom_G(N, \pi)$ if and only if $\tilde{f}$ satisfies

$$\tilde{f}(\sigma(g)(v)) = \pi(g)(\tilde{f}(v))$$

for all $g \in G/N$ and all $v \in V$.

If moreover N is a connected Lie group, $N/\overline{[N, N]}$ can be identified to $\mathbb{R}^n \times \mathbb{T}^k$ for some n, k. Consequently $V = \mathbb{R}^n$, and in this case, σ is a real finite dimensional representation (non unitary in general) of G/N. Hence the following ([5]):

Proposition 2.13. *Let N be a connected Lie group; $Hom_G(N,\pi)$ is isomorphic to the space of $\mathbb{R}$-linear maps from V to $\mathcal{H}_\pi$ which intertwine σ and π.*
If $(\sigma^{\mathbb{C}}, V^{\mathbb{C}})$ is the complexified representation obtained from (σ, V), the space $Hom_G(N,\pi)$ can be identified with the space of $\mathbb{C}$-linear maps from $V^{\mathbb{C}}$ to $\mathcal{H}_\pi$ which intertwine $\sigma^{\mathbb{C}}$ and π.

So we deduce:

Lemma 2.14. *Let π be a unitary irreducible representation of a connected Lie group N; then $Hom_G(N,\pi)$ does not vanish if and only if π is a subrepresentation of $\sigma^{\mathbb{C}}$. In particular there are only finitely many such representations and there are all of dimension at most $dim(\sigma) \leq dim(N/[N:N])$.*

In the case where N is a central subgroup, we have ([5]):

Lemma 2.15. *Let π a non trivial irreducible unitary representation of G and let C be a closed central subgroup of G. If $\overline{H^1}(G,\pi) \neq 0$, then $\pi|_C = 1$ and $\overline{H^1}(G,\pi) \cong \overline{H^1}(G/C,\dot{\pi})$.*

3 $\overline{H^1}(G,\pi)$ of connected amenable locally compact groups

3.1 Amenability and reduced-1 cohomology of unitary irreducible representations

In this section we will establish an analogue of Delorme's theorem (thm 1.1.) for connected amenable locally compact groups. More precisely, we will show that the reduced-1 cohomology of such a group is zero for all irreducible unitary representation except a finite number of finite dimensional ones.
We first establish the result for a connected amenable Lie group:

Theorem 3.1. *Let G be a connected amenable Lie group.*
Up to unitary equivalence, there are finitely many irreducible unitary representations π of G with $\overline{H^1}(G,\pi) \neq 0$. Moreover all these representations are finite dimensional and their dimensions are less than the (real) dimension of the radical of G.

Proof. Let π be an irreducible unitary representation of G such that $\overline{H^1}(G,\pi) \neq 0$.
First let us show that π is finite dimensional.
Consider the Lévi decomposition of G, RS, where R is the radical and S is semisimple (hence compact by amenability of G).
Claim: The restriction $\pi|_R$ has a finite dimensional subrepresentation.

Indeed, assume by contradiction that this is not the case. Then, as R is a connected solvable Lie group, $\overline{H^1}(R,\pi|_R) = 0$ by Delorme's theorem and proposition 2.7. Proposition 2.9 implies $\overline{H^1}(G,\pi) = 0$, contradicting our assumption. this proves the claim
Let χ be a finite dimensional subrepresentation of $\pi|_R$. Then we have $\pi|_R \otimes \overline{\chi} \supset 1$ which imply that $Ind_R^G(\pi|_R \otimes \overline{\chi}) = \pi \otimes Ind_R^G\overline{\chi} \supset \lambda_{G/R} = Ind_R^G 1$. But the quasi-regular representation $\lambda_{G/R}$ contains the trivial representation by compactness of G/R. So π must be finite dimensional.

Now let us show that there are only finitely many finite-dimensional irreducible representations of G with $\overline{H^1}(G,\pi) \neq 0$.

Let $\tilde{G}$ be the universal cover of G. If π is a unitary representation of G and if $\tilde{\pi}$ denotes the $\tilde{G}$-representation obtained by pulling π back , then $\overline{H^1}(G,\pi) \cong \overline{H^1}(\tilde{G},\tilde{\pi})$ (see [4]).
So we can assume G to be simply connected. The Lévi decomposition of G is then a semi-direct product $R \rtimes S$.
Let π be a finite dimensional irreducible unitary representation of G. By Lie's theorem, $\pi|_{[R,R]} = 1$ and because $[R,R]$ is a closed normal subgroup of G (see [8] Chap. XII Thm. 2.2), theorem 2.11 applies and gives

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/[R,R],\dot{\pi}) \oplus A(G,[R,R],\pi).$$

By lemma 2.14, $A(G,[R,R],\pi)$ is non zero only for finitely many representations π, of dimension at most $dim(R)$.
So we will show that $\overline{H^1}(G/[R,R],\dot{\pi})$ is non zero only for finitely many irreducible unitary representations.
By connectedness of R, $G/[R,R] = (\mathbb{R}^n \times \mathbb{T}^k) \rtimes S$ for some n,k. If $\dot{\pi}$ does not have non-zero $(\mathbb{R}^n \times \mathbb{T}^k)$-invariant vectors, then by proposition 2.9 and by the vanishing of the space $\overline{H^1}(\mathbb{R}^n \times \mathbb{T}^k,\sigma)$ for all unitary representations without non-zero invariant vectors (see [5]), we have $\overline{H^1}(G/[R,R],\dot{\pi}) = 0$.

If $\dot{\pi}$ has non-zero $(\mathbb{R}^n \times \mathbb{T}^k)$-invariant vectors, we get by irreducibility that $\dot{\pi}|_{(\mathbb{R}^n \times \mathbb{T}^k)} = 1$.

So by applying theorem 2.11 *i*):

$$\overline{H^1}(G/[R,R],\dot{\pi}) \cong \overline{H^1}(S,\dot{\pi}) \oplus A(G/[R,R],(\mathbb{R}^n \times \mathbb{T}^k),\dot{\pi}).$$

But S is compact, so $\overline{H^1}(S,\dot{\pi}) = 0$ and we apply lemma 2.14 to conclude that the space $A(G/[R,R],(\mathbb{R}^n \times \mathbb{T}^k),\dot{\pi})$ is non zero for only finitely many irreducible finite dimensional unitary representations, whose dimensions are less than the (real) dimension of the radical of G. ■

Example 3.2. *Let $G = \mathbb{C}^n \rtimes U(n)$ be the rigid motion group of $\mathbb{C}^n$; and let π be the unitary irreducible representation of G in $\mathbb{C}^n$ given by*

$$\pi(x,g) = g.$$

Define a cocycle in $Z^1(G,\pi)$ by setting $b(x,g) = x$. The corresponding affine action is the tautological one on the affine space underlying $\mathbb{C}^n$. This cocycle is not almost a coboundary, so $\overline{H^1}(G,\pi) \neq 0$.
This exemple shows that in the previous theorem the upper bound on the dimension of irreducible unitary representations with non vanishing reduced-1 cohomology, is optimal.

We will then use the well-known Montgomery-Zippin's theorem (see [13]):

Theorem 3.3. *(Montgomery-Zippin)*
Let G be a connected locally compact group. Then for every neighborhood of the neutral element V there exists a normal compact subgroup K_V of G contained in V, such that G/K_V is a real Lie group.

We then obtain

Theorem 3.4. *Let G a locally compact almost connected amenable group. The unitary irreducible representations with non vanishing reduced 1-cohomology are all finite dimensional and there are only finitely many such representations.*

Proof. By corollary 2.12, we can assume that G is connected. By theorem 3.3, there exists a normal compact subgroup K of G such that G/K is a Lie group. If π is a unitary irreducible representation of G, then:

i) Either $\pi|_K$ doesn't have non zero invariant vectors and then lemma 2.8 implies that $H^1(G,\pi) = 0$ which implies $\overline{H^1}(G,\pi) = 0$.

ii) Or $\pi|_K$ has non zero invariant vectors, and by irreducibility, $\pi|_K = 1$. So by corollary 2.12, $\overline{H^1}(G, \pi) = \overline{H^1}(G/K, \dot{\pi})$, and the previous theorem applies.

■

Recently, Y. Shalom introduced the property (H_{FD}) for a locally compact group (see [14]):
A locally compact group G has the property (H_{FD}) if for all irreducible representation π, $\overline{H^1}(G, \pi) \neq 0$ implies that π is finite dimensional. He shows in particular that this property is a quasi-isometry invariant among the class of finitely generated amenable groups.

Hence, a consequence of the preceding theorem is:

Corollary 3.5. *A locally compact almost connected amenable group has the property (H_{FD}).*

3.2 $\overline{H^1}(G, L^2(G))$ and amenability

In this section, we will prove the conjecture mentioned in the introduction for amenable connected locally compact groups. We will need the following preliminary lemma:

Lemma 3.6. *Let G be a locally compact group. If for all neighborhood V of the identity , there exists a normal compact subgroup K contained in V such that $\overline{H^1}(G/K, \lambda_{G/K}) = 0$, then $\overline{H^1}(G, \lambda_G) = 0$.*

Proof. Let $b \in Z^1(G, \lambda_G)$. For any compact normal subgroup K let us define a cocycle in $Z^1(G, L^2(G)^K)$ (where $L^2(G)^K$ is the space of (right) K-invariant vectors in $L^2(G)$) by:

$$(b^K(g))(h) = \int_K b(g)(hk)\, dk$$

(dk is the normalized Haar measure on K). So we have:

$$\| b^K(g) - b(g) \|_2^2 \;=\; \int_G | b^K(g)(h) - b(g)(h) |^2\, dh$$

$$
\begin{aligned}
&= \int_G | \int_K (b(g)(hk) - b(g)(h))dk |^2 \, dh \\
&\leq \int_G \int_K | b(g)(hk) - b(g)(h) |^2 \, dkdh \\
&= \int_K \int_G | b(g)(hk) - b(g)(h) |^2 \, dhdk \\
&= \int_K \| \rho(k)b(g) - b(g) \|_2^2 \, dk.
\end{aligned}
$$

Finally as the right regular representation ρ is strongly continuous at the neutral element, there exists for every $\varepsilon > 0$, every compact subset Q of G, a neighborhood V of e such that $\| \rho(k)b(g) - b(g) \|_2 \leq \varepsilon$, $\forall g \in Q$, $\forall k \in V$. We easily conclude by using the cohomological assumption.

■

Theorem 3.7. *Let G be a locally compact separable almost connected group. If G is amenable, then $\overline{H^1}(G, L^2(G)) = 0$.*

Proof. Let us recall that if N is a closed subgroup of G, then $\lambda_G|_N = [G : N] \cdot \lambda_N$ and so $\overline{H^1}(N, \lambda_G|_N) = 0 \Leftrightarrow \overline{H^1}(N, \lambda_N) = 0$ (see e.g. [12]). So by proposition 2.9, we can replace G by its connected component of 1; i.e. we can assume that G is connected and non compact.

By theorem 3.3, for every neighborhood V of the identity in G, there exists a compact normal subgroup K_V, such that G/K_V is a Lie group. So G/K_V is an amenable connected Lie group. Since G/K_V is non-compact a finite set of finite dimensional representations cannot appear discretely in the direct integral decomposition into irreducible representations of the regular representation of G/K_V. So by theorem 3.4, $\overline{H^1}(G/K_V, \lambda_{G/K_V}) = 0$ and by lemma 3.5, $\overline{H^1}(G, \lambda_G)$ must vanish. ■

4 $\overline{H^1}(G, \pi)$ and the Haagerup property

In [3], the authors classify connected Lie groups having the Haagerup property. They show that such a group is necessarily locally isomorphic to a product $M \times SO(n_1, 1) \times ... \times SO(n_k, 1) \times SU(m_1, 1) \times ... \times SU(m_l, 1)$, where M is an amenable Lie group. By using Delorme's theorem [4] on the 1-cohomology of the groups $SO(n,1)$ and $SU(m,1)$, we will classify the irreducible unitary representations of a connected group having Haagerup property that give rise to non zero first reduced cohomology space.

Delorme's theorem that we will need is the following:

Theorem 4.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$. Then there exists at least one irreducible unitary representation and at most two with non trivial 1-cohomology. Moreover, these representations are infinite dimensional.*

From this and the previous theorem, we deduce:

Theorem 4.2. *Let G be a connected Lie group with Haagerup property. There are finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G,\pi)$.*

Proof. As in the proof of theorem 3.1, we can assume G to be simply connected. Because G has the Haagerup property, it is isomorphic to a product ([3], thm 4.0.1)

$$M \times \widetilde{SO(n_1,1)} \times ... \times \widetilde{SO(n_k,1)} \times \widetilde{SU(m_1,1)} \times ... \times \widetilde{SU(m_l,1)}.$$

where M is amenable.
Let us show the result by induction on the number of factors in the preceding direct product. If there is only one factor, then the result follows from theorem 3.1 and 4.1. Let us assume that there are n factors in the direct product decomposition of G and let π be an irreducible unitary representation of G.

If π doesn't have non zero invariant vectors for each factors, then $\overline{H^1}(G,\pi) = 0$ (see [15]). If π has a non zero invariant vector by at least one factor, set N to be the product of those factors where π has non zero invariant vectors. By, $\pi|_N = 1$ so by theorem 2.11

$$\overline{H^1}(G,\pi) = \overline{H^1}(G/N,\dot{\pi}) \oplus Hom_G(N,\pi).$$

then we conclude, by the induction assumption and lemma 2.14. ∎

Again by using theorem 3.3, we obtain a similar result for connected locally compact groups having the Haagerup property.

Theorem 4.3. *Let G be a almost connected locally compact group with the Haagerup property. There are finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G, \pi)$.*

Proof. Argue similarly as in the proof of theorem 3.4. ∎

5 $\overline{H^1}(G, \pi)$ and the relative property (T)

In this section we will study 1-cohomology and the reduced-1 cohomology with values in an irreducible unitary representation of a locally compact group G having a closed normal subgroup N such that the pair (G, N) has the relative property (T).

Proposition 5.1. *Let G be a locally compact and N a closed normal subgroup such that (G, N) has relative property (T). Let π be an irreducible unitary representation of G. We have the following alternative :*

i) either $\pi|_N$ does not have non zero invariant vectors, and then $H^1(G, \pi) = \overline{H^1}(G, \pi) = 0$;

ii) or $\pi|_N = 1$ and we have the isomorphisms $H^1(G, \pi) \cong H^1(G/N, \dot{\pi})$, $\overline{H^1}(G, \pi) \cong \overline{H^1}(G/N, \dot{\pi})$.

Proof. By definition of relative property (T), the restriction map Res : $H^1(G, \pi) \to H^1(N, \pi|_N)$ is identically zero. So if $\pi|_N$ does not have non zero invariant vectors, $H^1(G, \pi) = 0$, by lemma 2.8.
If $\pi|_N$ has non zero invariant vectors then by irreducibility, $\pi|_N = 1$, and we apply theorem 2.11 to get the isomorphisms:

$$H^1(G, \pi) \cong H^1(G/N, \pi) \oplus Im(Res : H^1(G, \pi) \to H^1(N, 1))$$

$$\overline{H^1}(G, \pi) \cong \overline{H^1}(G/N, \pi) \oplus Im(Res : H^1(G, \pi) \to H^1(N, 1)).$$

But by the relative property (T), the second summand is zero. ∎

6 $\overline{H^1}(G,\pi)$ of locally compact connected groups

To study reduced-1 cohomology of connected locally compact groups, we will investigate first the case of connected Lie groups, and then apply the "Montgomery-Zippin" argument of the previous sections. Let us recall the following theorem ([3], thm 4.0.1)

Theorem 6.1. *Let G be a non compact connected Lie group. Then either G has the Haagerup property, or there exists a closed non compact connected normal subgroup N such that the pair (G,N) has the relative property (T) (these properties are mutually exclusive).*

Remark 6.2. *In [3] theorem 4.0.1, when G does not have the Haagerup property, it is not mentioned that the closed subgroup N such that (G,N) has the relative property (T) is normal and connected. But looking closely at the proof (section 4.1.3.) shows that the constructed subgroup is indeed normal and connected.*

We obtain:

Theorem 6.3. *Let G be a connected Lie group. Then there are only finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G,\pi)$.*

Proof. If G is compact then it has property (T) and the result is clear. Assume that G is non-compact and let us show the result by induction on the dimension of G. If G has the Haagerup property, we use theorem 4.2. If it is not the case, by theorem 6.1, there exists a non compact connected closed normal subgroup N such that (G,N) has the relative property (T). By proposition 5.1, the irreducible unitary representations π of G that have non vanishing reduced-1 cohomology satisfy $\pi|_N = 1$ and then proposition 5.1 applies to give :

$$\overline{H^1}(G,\pi) \cong \overline{H^1}(G/N,\pi).$$

We conclude by using the induction hypothesis. ■

More precisely, we have:

Theorem 6.4. *Let G be a almost connected locally compact group. Then there are only finitely many irreducible unitary representations with non vanishing $\overline{H^1}(G,\pi)$.*
Moreover, if G does not have property (T) (which implies the existence of

an irreducible unitary representation π of G with $\overline{H^1}(G,\pi) \neq 0$), any such non trivial representation π factors through an irreducible unitary representation σ of a group H isomorphic to $PO(n,1), PU(m,1)$ or to a non-compact amenable non-nilpotent group H such that $\overline{H^1}(H,\sigma) \cong \overline{H^1}(G,\pi) \neq 0$.

Proof. The first statement is a direct consequence of the theorem 6.3 and the argument used in the proof of the theorem 3.4. If G does not have the property (T), then the existence of a irreducible unitary representation with non vanishing $\overline{H^1}(G,\pi)$ is given by proposition in [15].

If the only irreducible representation having non trivial reduced-1 cohomology is the trivial representation there is nothing to prove. Let π be a non trivial unitary representation of G with $\overline{H^1}(G,\pi) \neq 0$. By theorem 3.3, there exists a compact normal subgroup K of G such that $G_0 \doteq G/K$ is a Lie group. As $\overline{H^1}(G,\pi) \neq 0$, π factors through G_0 and $\overline{H^1}(G,\pi) \cong \overline{H^1}(G_0,\dot{\pi}) \neq 0$.

Claim: There exists a non compact closed connected subgroup N such that $G_N \doteq G_0/N$ has the Haagerup property, $\pi|_N = 1$, and $\overline{H^1}(G_N,\pi) \cong \overline{H^1}(G_0,\pi) \neq 0$.
Indeed if G_0 has the Haagerup property, we end here. If not there exists a closed connected normal subgroup N_0 of G_0 such that (G_0, N_0) has relative property (T). By proposition 5.1, $\pi|_{N_0} = 1$ and $\overline{H^1}(G_0/N_0,\pi) \cong \overline{H^1}(G_0,\pi) \neq 0$. If G_0/N_0 has the Haagerup property, we are done. If not as G_0/N_0 doesn't have property (T), we apply again the same arguments. As the dimension of the Lie group strictly decrease at each step(N_0 is connected), the procedure ends and the final quotient cannot have property (T), and in particular is not compact. This prove the claim.
Let $\widetilde{\pi}$ be the representation defined canonically on the universal cover $\widetilde{G_N}$ of G_N. By [4], $\overline{H^1}(\widetilde{G_N},\widetilde{\pi}) \cong \overline{H^1}(G_N,\pi) \neq 0$. Applying the classification theorem of [3], $\widetilde{G_N}$ is a product of (simply connected) groups $\widetilde{SO(n,1)}$, and/or $\widetilde{SU(n,1)}$ and/or amenable groups. By proposition 3.2 of [15], π is trivial on at least one factor. But then, by proposition 2.13 (applied to $\sigma^{\mathbb{C}} = 1$), and as $\widetilde{\pi}$ is not trivial, $\widetilde{\pi}$ is trivial on all factors except one, that we will denote by $\widetilde{H}$. Moreover we have $\overline{H^1}(G,\pi) \cong \overline{H^1}(\widetilde{H},\widetilde{\pi}) \neq 0$. However, $\widetilde{\pi}$ is trivial on the center of $\widetilde{H}$. So if we denote by H the quotient $\widetilde{H}/\mathcal{Z}(\widetilde{H})$, we have that $\overline{H^1}(H,\widetilde{\pi}) \cong \overline{H^1}(\widetilde{H},\widetilde{\pi}) \neq 0$. Notice that as $\widetilde{\pi}$ is irreducible and non trivial, H cannot be nilpotent.
By construction, G maps onto H, π is trivial on the kernel of this surjection,

and $\pi = \widetilde{\pi}$ on H. So $\overline{H^1}(G_N, \pi) \cong \overline{H^1}(H, \pi) \neq 0$ and by construction, H is isomorphic to either $PO(n, 1)$ or $PU(n, 1)$ or an amenable group. ■

Remark 6.5. *There is no analogue of the theorem 6.5 for non-connected groups. To see it, consider the free group $G = \mathbb{F}_2$ on 2 generators. A.Guichardet [5] observed that $H^1(G, \pi) \neq 0$ for every unitary representation π of G. Now, if π is finite dimensional, we even have $\overline{H^1}(G, \pi) \neq 0$. In particular, for every character χ of G, $\overline{H^1}(G, \chi) \neq 0$, so we get a continuum of irreducible representations carrying reduced 1-cohomology.*

7 Application to harmonic analysis

Let G be a connected unimodular Lie group and let (M, ν) a smooth non compact connected manifold on which G acts transitively by diffeomorphisms and respecting a measure (σ-finite) ν. If μ is a probability measure on G, we say that a smooth function f on M is μ-harmonic if $f(x) = \int_G f(q^{-1} \cdot x)\, d\mu(q)$ (where $\cdot$ denote the action of G on M).
Recall that if $(X_1, \dots, X_n)$ is a Hörmander system of smooth G-invariants vector fields (i.e. a family of smooth vector fields such that the Lie algebra they generates is the whole tangent space at each point), the gradient of a function $f \in \mathcal{C}^\infty(M)$ is defined by $\nabla f = (X_1 f, \dots, X_n f)$ and that $|\nabla f| = (\sum_{i=1}^{n} |X_i f|^2)^{\frac{1}{2}}$.
On G, a G-invariant (for the right multiplication) Hörmander system always exists. Consequently as G acts transitively by diffeomorphisms, we obtain a G-invariant Hörmander system on M. Fix once and for all a Hörmander system on G.
Finally, $f \in \mathcal{C}^\infty(M)$ is said to be Dirichlet finite, if
$||\nabla f||_{L^2(M,\nu)} < \infty$. We will denote by π the action of G on $\mathcal{C}^\infty(M)$ defined by $\pi(g) f(x) = f(g^{-1} \cdot x)$.

With these definitions and notations, we will establish in this section a link between the existence of Dirichlet-finite functions on M and the reduced-1 cohomology of G with values in $L^2(M)$.
First some technical lemmas

Lemma 7.1. *Let M be a manifold, $(X_1, \ldots, X_n)$ a Hörmander system and $\gamma : [0,a] \to M$ a differentiable path on M tangent to the Hörmander system with $\|\gamma'(t)\|_2 \leq 1$. For $f \in \mathcal{C}^\infty(M)$, we have the following inequality:*

$$|f(\gamma(a)) - f(\gamma(0))| \leq \int_0^a |\nabla f(\gamma(t))| dt.$$

Proof. For $t \in [0,a]$ we have:

$$\begin{aligned} |f(\gamma(a)) - f(\gamma(0))| &= |\int_0^a \frac{d}{dt} f(\gamma(t))\, dt| \\ &\leq \int_0^a |df_{\gamma(t)}(\gamma'(t))|\, dt. \end{aligned}$$

Moreover if we write $\gamma'(t) = \sum_{i=0}^{k} a_i(t) X_i(\gamma(t))$, we have as $df_{\gamma(t)}(X_i(\gamma(t))) = X_i f(\gamma(t))$, using the Cauchy-Schwartz inequality:

$$\begin{aligned} |df_{\gamma(t)}(\gamma'(t))| &= |\sum_{i=0}^{k} a_i(t) X_i f(\gamma(t))| \\ &\leq \|\gamma'(t)\| |\nabla f(\gamma(t))| \\ &\leq |\nabla f(\gamma(t))| \end{aligned}$$

Hence the claimed inequality. ■

Lemma 7.2. *Let f be a smooth Dirichlet finite function on M. Then for all $h \in G$, there exists $a = a(h) > 0$ such that*

$$\|\pi(h)f - f\|_{L^2(M,\nu)} \leq a \cdot ||\nabla f||_{L^2(M,\nu)}.$$

Proof. Let $h \in G$ and let $\gamma : [0,a] \to G$ be an absolutely continuous path such that $\gamma(0) = e$, $\gamma(a) = h$, and $\gamma'(t) = \sum_{i=1}^{n} a_i(t) X_i(\gamma(t))$ a.e. with

$\sum_{i=1}^{n} a_i^2(t) \leq 1$ (it always exists, see [16] III.4). As the action is smooth and the Hörmander system is invariant, we apply the preceding lemma to the path $t \mapsto \gamma(t)^{-1} \cdot x$ and we get:

$$|f(h^{-1} \cdot x) - f(x)| \leq \int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)| dt.$$

So by Cauchy-Schwarz, $|f(h^{-1} \cdot x) - f(x)|^2 \leq a \int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)|^2 dt$.
Let $\overline{(K_n)_{n\geq 1}}$ be an increasing sequence of compact subsets of M such that $\bigcup_{n\geq 1} K_n = M$.For all n we have:

$$\begin{aligned}
\int_{K_n} |f(h^{-1} \cdot x) - f(x)|^2 dx &\leq a \int_{K_n} \int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)|^2 dt d\nu(x) \\
&\leq a \int_M \int_0^a |\nabla f(\gamma(t)^{-1} \cdot x)|^2 dt d\nu(x) \\
&= a \int_0^a \int_M |\nabla f(\gamma(t)^{-1} \cdot x)|^2 d\nu(x) dt \\
&= a \int_0^a \int_M |\nabla f(x)|^2 d\nu(x) dt \\
&= a^2 \|\nabla f\|_2^2.
\end{aligned}$$

hence we conclude that $\|\pi(h)f - f\|_2 \leq a \cdot ||\nabla f||_2$. ■

Here is the main theorem of this section:

Theorem 7.3. *Let G be a connected unimodular Lie group acting smoothly and transitively on a non-compact connected smooth manifold M endowed with a G-invariant (σ-finite) measure ν and let μ be a probability measure on G with compact symmetric support generating G.*
If $\overline{H^1}(G, L^2(M,\nu)) = 0$, then every Dirichlet-finite μ-harmonic smooth function on M is constant.

Proof. Set $L^2(M) = L^2(M,\nu)$ and let $\mathcal{D}(M)$ be the following quotient space: $\{f \in \mathcal{C}^\infty(M) \mid ||\pi(g)f - f||_2 < \infty \, \forall g \in G\}/\mathbb{C}$.
Consider the pre-Hilbert structure on $\mathcal{D}(M)$ given by $||f||^2_{\mathcal{D}(M)} = \int_G ||\pi(q)f - f||^2_{L^2(M)} d\mu(q)$. Notice that $\mathcal{D}(M)$ is Hausdorff because $||f||^2_{\mathcal{D}(M)} = 0$ iff

$\pi(g)f = f$, $\forall g \in supp(\mu)$, which is equivalent to $\pi(g)f = f$, $\forall g \in G$ (because $supp(\mu)$ generates G) and which is also equivalent to the fact that f is constant (this follows from the transitivity of the action).
Let i be the canonical embedding of $\mathcal{C}^\infty(M) \cap L^2(M)$ in $\mathcal{D}(M)$. For all $f \in \mathcal{D}(M)$, denote by $\theta(f)$ the algebraic cocycle given by $g \mapsto \pi(g)f - f$. This cocycle is weakly measurable, so by [5], it is continuous for the topology of uniform convergence on compact subsets (we use here the fact that G is separable).
By assumption $\theta(f)$ is almost a coboundary. As $\mathcal{C}^\infty(M) \cap L^2(M)$ is $|| \, ||_2-$dense in $L^2(M)$, there exists a sequence $(\xi_n)_{n\geq 1}$ in $\mathcal{C}^\infty(M) \cap L^2(M)$ such that $\theta(f)(g) = \lim_{n\to\infty} \pi(g)\xi_n - \xi_n$ uniformly on compact subsets of G. Hence $\int_G ||\pi(q)(f-\xi_n) - (f-\xi_n)||^2_{L^2(M)} d\mu(q) \overset{n\mapsto\infty}{\longrightarrow} 0$ since μ has compact support.
This shows that $i(\xi_n) \overset{n\mapsto\infty}{\longrightarrow} f$ in $\mathcal{D}(M)$. In other words, $i(\mathcal{C}^\infty(M) \cap L^2(M))$ is dense in $\mathcal{D}(M)$.
So $i(\mathcal{C}^\infty(M) \cap L^2(M))^\perp = 0$, because $\mathcal{D}(M)$ is Hausdorff.

Let us compute this orthogonal complement :

$$
\begin{aligned}
& f \in i(\mathcal{C}^\infty(M) \cap L^2(M))^\perp \\
\Leftrightarrow\quad & \int_G < \rho(q)f - f \,|\, \rho(q)\xi - \xi >_2 d\mu(q) = 0 \ , \ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \\
\Leftrightarrow\quad & \int_G < \rho(q)f - f \,|\, \rho(q)\xi >_2 d\mu(q) - \int_G < \rho(q)f - f \,|\, \xi >_2 d\mu(q) = 0 \ , \\
& \qquad \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \\
\Leftrightarrow\quad & \int_G < f - \rho(q^{-1})f \,|\, \xi >_2 d\mu(q) - \int_G < \rho(q)f - f \,|\, \xi >_2 d\mu(q) = 0 \ , \\
& \qquad \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \\
\Leftrightarrow\quad & -2\int_G < \rho(q)f - f \,|\, \xi >_2 d\mu(q) = 0 \ , \ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) (\text{as } \mu \text{ is symmetric }) \\
\Leftrightarrow\quad & < \int_G (\rho(q)f - f) d\mu(q) \,|\, \xi >_2 = 0 \ , \ \forall \xi \in \mathcal{C}^\infty(M) \cap L^2(M) \\
\Leftrightarrow\quad & \int_G (\rho(q)f - f) d\mu(q) = 0 \\
\Leftrightarrow\quad & \int_G \rho(q) f d\mu(q) = f \\
\Leftrightarrow\quad & \int_G f(q^{-1} \cdot x) d\mu(q) = f(x) \ , \ \forall x \in M
\end{aligned}
$$

So the orthogonal complement of $i(\mathcal{C}^\infty(M) \cap L^2(M))$ is nothing else than the space of μ-harmonic functions in $\mathcal{D}(M)$.

Now, let f be a smooth Dirichlet finite function. By the preceding lemma,

$$||\pi(g)f - f||_{L^2(M)} \leq a(g) ||\nabla f||_{L^2(M)}, \ \forall g \in G.$$

So such a f is (modulo constant functions) in $\mathcal{D}(M)$. So if f is μ-harmonic and Dirichlet finite, then it is constant. ■

We get immediately the following corollary

Corollary 7.4. *Let G be a connected Lie group having property (T). If G acts smoothly and transitively on a non-compact connected smooth manifold M endowed with a G-invariant (σ-finite) measure ν and if μ is a probability measure on G with compact symmetric support generating G, then a Dirichlet-finite μ-harmonic smooth function on M is constant.*

In the case where G acts by translation on itself, we obtain immediately:

Corollary 7.5. *Let G be a connected unimodular Lie group such that $\overline{H^1}(G, L^2(G)) = 0$ and let μ be a probability measure on G with compact symmetric support generating G.*
Then a Dirichlet-finite μ-harmonic smooth function on G is constant.

By theorem 3.6, we also have

Corollary 7.6. *Let G be a amenable connected unimodular Lie group and let μ be a probability measure on G with compact symmetric support generating G.*
Then a finite Dirichlet μ-harmonic smooth function on G is constant.

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