Some remarks on harmonic functions on homogeneous graphs

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Abstract

We obtain a new result concerning harmonic functions on Cayley graphs X: either every nonconstant harmonic function has large radial variation (in a certain sense), or there is a nontrivial hyperbolic boundary at infinity of X. In the latter case, relying on a theorem of Woess, it follows that the Dirichlet problem is solvable with respect to this boundary. Certain relations to group cohomology are also discussed.

1 Introduction

Let X be an connected, locally finite, infinite, oriented graph with no loops and with countable vertex set $V(X)$ and edge set $E(X)$. Two vertices x and y are *adjacent*, denoted $x \sim y$, if they are connected by an edge, and denote by $\deg(x)$ the number of neighbours of x. The *combinatorial laplacian* acting on $\{f : V(X) \to \mathbb{R}\}$ is

$$(\Delta f)(x) = \deg(x) f(x) - \sum_{y \sim x} f(y).$$

This yields a notion of *harmonic functions* on X, namely those f satisfying

$$\Delta f \equiv 0,$$

which amounts to a mean value property for f.

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The laplacian and corresponding harmonic functions are basic and important objects associated to a graph. As in classical potential theory it is intimately connected with an associated *random walk*, namely, the Markov chain defined by transition probabilities

$$P(x \to y) = \frac{1}{\deg(x)}$$

whenever $y \sim x$ and 0 otherwise. For example, suppose that X is compactified with a boundary ∂X (see section 2) and that almost every trajectory of the random walk converges to some point in ∂X. Then a *Poisson formula* gives harmonic functions h on X from any boundary value $f \in L^\infty(\partial X, \mathbb{R})$:

$$h(x) := \int_{\partial X} f(\xi) d\lambda_x(\xi),$$

where λ_x are the exit measures from the converging random trajectories starting at x.

These consideration are for example relevant to electrical network theory: Let the edges correspond to 1 Ohm resistances. Then by Kirchoff's laws the *passive currents* are exactly the gradients df for f harmonic belonging to the space of *Dirichlet finite functions*

$$\{f : V(X) \to \mathbb{R} \text{ such that } df \in l^2(E(X), \mathbb{R})\},$$

(which physically means finiteness of energy). We refer to Soardi's book [So 94] for a nice mathematical exposition on this topic.

The main result in the present note is the following:

Theorem 1 *Let $X = X(\Gamma, S)$ be the Cayley graph of a group Γ generated by a finite set S. Either*

$$\sum_{r>0} \sup_{|x|\geq r} |dh(x)| = \infty$$

for every nonconstant harmonic function h, or X has a nontrivial boundary $\partial_F X$. In the latter case, the Dirichlet problem is solvable with respect to $\partial_F X$, which moreover (together with a suitable measure) can be identified with the Poisson boundary.

This theorem is the combination of Theorems 2 and 5 below. Groups for which the first alternative in the theorem holds include groups not containing a free noncommutative subgroups, $SL_n(\mathbb{Z})$, $n \geq 3$, as well as most

Artin braid groups and mapping class groups. Groups for which the second alternative in the theorem holds include groups with infinitely many ends, nonelementary hyperbolic groups and nonelementary geometrically finite Kleinian groups.

2 Hyperbolic compactifications of infinite graphs

We consider X as a geodesic metric space (X, d) in a standard way: we realize X geometrically by for each edge assign a copy of the unit interval with the standard metric, glue them to their associated vertices and let d denote the induced path metric on all of X. Fix a base point $x_0 \in X$. Let $|x| = d(x_0, x)$ and $|A| = \inf_{a \in A} d(x_0, a)$.

We now describe the compactification of X (basically) following [Fl 80]. Let F be a *Floyd admissible function*, i.e. a monotonically decreasing function $\mathbb{N} \to \mathbb{R}_{>0}$ which is summable:

$$\sum_{j=0}^{\infty} F(r) < \infty.$$

Assume in addition that there is a constant $L > 0$ such that $F(r+1) \leq F(r) \leq LF(r+1)$ for all r. The new length of an edge connecting x and y is

$$F(|\{x, y\}|)$$

(instead of 1). A *path* is a sequence of vertices $\{x_i\}$ such that every x_i and x_{i+1} are adjacent. We define the d_F-length L_F of a path $\alpha = \{x_i\}$ in the graph:

$$L_F(\alpha) = \sum_i F(|\{x_i, x_{i+1}\}|)$$

and the new distance is

$$d_F(z, w) := \inf_{\alpha} L_F(\alpha),$$

where the infimum is taken over all paths α connecting z and w. It is straightforward to verify that d_F satisfies the axioms of a metric. In particular, since (X, d) is a geodesic space any two points z, w can be joined by a geodesic β, so $d'(z, w) \leq L_f(\beta) < \infty$. (When we speak about geodesics it will always refer to the distance d.) Note also that X has finite d_F-diameter because F is summable.

We now define $\overline{X}^F$ to be the completion of (X, d_F) in the sense of metric spaces and the boundary is $\partial_F X = \overline{X}^F \setminus X$, which we refer to as a *Floyd* *boundary of* X. It is simple to see that this completion is compact, see [Fl 80].

The Cayley(-Dehn) graph of a group. Let Γ be an infinite, finitely generated group. Choose a finite generating set S and fix the corresponding Cayley graph $X = X(\Gamma, S)$: the vertex set is Γ and the oriented edges are $[x, xs]$ where $s \in S$ and $x \in \Gamma$.

A remark on an alternative compactification. One can instead of choosing one F as above, take all such functions $\mathcal{F}$ bounded by 1 (or some larger class of bounded functions) and construct the corresponding Stone-Cech compactification by embedding X via the evaluation map to the Cartesian product $[0,1]^{\mathcal{F}}$. Compare with [R91] or [Gr 93, Ch. 8].

3 The Dirichlet problem

One says that the Dirichlet problem is solvable if every continuous function on ∂X has a continuous extension to $\overline{X}$ which is harmonic on X.

Let X be as in the introduction and assume that $|\partial_F X| > 2$ for some Floyd-admissible F. It is established in [K 02] that the convergence and projectivity axioms of Woess are satisfied and that the action of $Aut(X)$ extends continuously to an action by homeomorphism of $\overline{X}^F$. In other words, $\overline{X}^F$ is a contractive $Aut(X)$-compactification in the sense of [Wo 00].

Assume now that X is a Cayley graph as in section 2 of an infinite, finitely generated group Γ. Since it is clear that Γ acts transitively on itself by translation on the left as graph automorphisms and moreover it is proved in [K 02] that if $|\partial_F X| > 2$ then $|\partial_F X| = \infty$ and that Γ does not have a global fixpoint in $\partial_F X$. We can therefore invoke Woess' Theorem 20.13 in [Wo 00] and [K 01] (which uses work of Kaimanovich) to obtain that:

Theorem 2 *Let $X = X(\Gamma, S)$ be the Cayley graph of a group Γ generated by a finite set S and assume that $\partial_F X$ is a Floyd boundary containing more than 2 points. Then the Dirichlet problem with respect to $\overline{X}^F$ is solvable and $\partial_F X$ with the induced harmonic measure is isomorphic to the Poisson boundary.*

This generalizes previous works on the space of ends and the standard boundary of Gromov hyperbolic graphs, see the notes and references in [Wo 00].

In order to give some intuition behind this theorem we give a simple argument in a special case to prove how random trajectories converg: Let

$F(r) = 1/(1+r^2)$ for simplicity and suppose y_n is a sequence of points in X such that $d(y_n, y_{n+1}) = 1$ and $d(x_0, y_n) \geq An$ for some $A > 0$. Then since it is readily checked that

$$d_F(y_n, y_m) \leq \sum_{n \leq k \leq m} \frac{1}{1 + (Ak)^2}$$

it follows that y_n is a d_F-Cauchy sequence and hence converges to a point in $\partial_F X$.

4 Constructing hyperbolic compactification from certain harmonic functions

We want to construct a compactification as in the previous section starting from a harmonic function of a certain type and begin by recalling the maximum principle:

Proposition 3 *Suppose that h is a harmonic function such that $h(x) \to 0$ as $x \to \infty$. Then $h \equiv 0$.*

Proof. Since h tends to 0 at infinity we may assume it has a maximum (or minimum) at some point x. But then by the mean value property ($\Delta h \equiv 0$) it holds for all neighbours y of x that $h(x) = h(y)$. As X is connected this equation propagates everywhere and we get that h must be constant. ■

Given a function $f : \Gamma \to \mathbb{R}$ the differential df is the function on $E(X)$ defined by

$$df([x, xs]) = f(xs) - f(x).$$

Suppose that h is a nonconstant harmonic function such that

$$\sum_{r>0} \sup_{|x| \geq r} |dh(x)| < \infty.$$

Note that it follows that

$$\lim_{n \to \infty} h(\gamma(n))$$

exists for any geodesic ray γ.

The preliminary candidate for a Floyd admissible function is

$$H(r) := \sup_{d(x, x_0) \geq r} |dh(x)|,$$

since this function is clearly monotonically decreasing, positive and summable. It remains to prove the existence of $L > 0$ such that $H(r+1) \leq H(r) \leq LH(r+1)$ for all r. This technical property, or something similar, is used to show that the isometric action of Γ on the Cayley graph continuously extends to an action by homeomorphisms of the completion $\overline{X}^H$, cf. [K 02]. To guarantee this property we modify H as given in the following technical lemma:

Lemma 4 *There is a summable function F such that $F(r) \geq H(r)$ and $F(r+1) \leq F(r) \leq 2F(r+1)$ for every r.*

Proof. Let r_i be times when

$$H(r_i) = \varepsilon_i H(r_i - 1)$$

and $\varepsilon_i < 1/2$. If these occur often, say with positive density then H decreases exponentially, and we can for example set $F(r) = H(r) + C/(1 + r^\alpha)$ for a suitable constant $C > 0$ and $\alpha > 1$. Otherwise, if H decreases a lower than some $1/(1 + r^\alpha)$, for some $\alpha > 0$, then we can at a time r_i let $F(r_i + k) = \frac{1}{2^k} H(r_i - 1)$ for $k \geq 0$ as long as it is larger than $H(r_i + k)$. Because of the slow growth of H the times r_i occur seldom (the more seldom, the smaller ε_i), which implies that the new F, which most of the time is equal to H, is summable. ■

Now we show that $|\partial_F X| \geq 2$. To see this note that for any (edge) path connecting two points $x, y \in X$ we have

$$\begin{aligned} |h(x) - h(y)| &= |h(x) - h(x_1) + h(x_1) - h(x_2) + ... - h(y)| \\ &\leq \sum_i |h(x_i) - h(x_{i+1})| = \sum_i H(|\{x_i, x_{i+1}\}|) \\ &\leq L_F(\{x_i\}). \end{aligned}$$

Since this estimate holds for any path $\{x_i\}$ connecting x and y we get that

$$|h(x) - h(y)| \leq d_F(x, y).$$

Therefore, as f is a nonconstant harmonic functions it has at least two distinct limiting values (Proposition 3), there are two geodesic rays which converge to (end at) two different points in $\partial_F \Gamma$.

5 Radial variation

The radial variation of (ordinary) harmonic functions were studied by Fatou, Zygmund, Rudin, Bourgain and others. See [CFPR 01] and [KW 00] for analogous results for harmonic functions on trees. The following result is perhaps of a new type for harmonic functions on graphs/groups:

Theorem 5 *Assume Γ is an infinite, finitely generated group not admitting an infinite Floyd boundary. If h is a harmonic function on Γ (with respect to some generators) such that*

$$\sum_{r>0} \sup_{|x|\geq r} |dh(x)| < \infty,$$

then h is constant.

Proof. It is well known that if $|\partial_F X| = 2$ then Γ is virtually $\mathbb{Z}$ and it does not admit any nonconstant harmonic functions. In view of section 4 the theorem now follows. ■

The above result can be viewed as an anti-Fatou-type or Liouville-property statement.

In [K 02], it is proved that if Γ does not contain a free nonabelian subgroup then every $\partial_F \Gamma$. Moreover groups, such as $SL_n(\mathbb{Z})$, for $n \geq 3$, most Artin braid groups, mapping class groups and automorphism groups of free groups, are shown in [KN 02] not to admit any nontrivial Floyd boundary. Same is true if Γ has an infinite, amenable (or, more generally, does not contain a nonabelian free subgroup) normal subgroup. On the other hand it is known that any nonamenable group admits many nonconstant bounded harmonic functions. We may therefore for example formulate:

Theorem 6 *Let Γ be $SL_n(\mathbb{Z})$ for $n \geq 3$ or a group containing an infinite, amenable normal subgroup. Choose some finite generating set S. Then for any nonconstant harmonic function h it holds that*

$$\sum_{r>0} \sup_{|x|\geq r} |dh(x)| = \infty.$$

6 Some group cohomology

It is possible to relate Floyd admissible functions which gives nontrivial boundaries to a certain reduced first cohomology group: Let L^1C_0 denote the functions on Γ whose supremum outside balls is integrable as in section 4. Define moreover

$$\overline{L^1C_0(X)} = \{f \in C_0(X) : df \in L^1C_0(E(X)\}$$

and finally

$$\overline{H}^1(\Gamma, L^1C_0) \cong \{f : \Gamma \to \mathbb{R} \text{ such that } df \in L^1C_0(E_\Gamma)\}/(\mathbb{R}+\overline{L^1C_0(\Gamma)}).$$

(This is just one possible way of defining these objects.)

Theorem 7 *Let Γ be an infinite, finitely generated group. The maximal number of points in a Floyd boundary of Γ equals*

$$1 + \dim \overline{H}^1(\Gamma, L^1C_0).$$

This is in analogy with a standard result essentially due to Specker [Sp 49]:

Theorem 8 *Let Γ be an infinite, finitely generated group. The number of ends equals*

$$1 + \dim H^1(\Gamma, C_c).$$

(Here C_c denotes the finitely supported real valued functions on a finitely generated group Γ, also called the group algebra and denoted $\mathbb{R}\Gamma$. Note also that as in the previous theorem, by "equal" is meant that, if both numbers are finite then there are equal, otherwise both are infinite.) As expected,

$$H^1(\Gamma, C_c) \rightarrowtail \overline{H}^1(\Gamma, L^1C_0)$$

in a natural way. These statements can be proved by making sutiable modifications of [BV 97].

Since L^2-cohomology (see [CG 86] and [BV 97]) is particularly important (for example for the theory of electrical networks), it make sense to raise:

Question. Given a finitely generated infinite group Γ with nontrivial first L^2-Betti number, what properties does the action of Γ on the associated compactification have?

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