On Property (T) for Pairs of Topological Groups *

Paul Jolissaint

April 7, 2005

1 On a theorem of Bates and Robertson

In 1967, D. Kazhdan defined Property (T) for locally compact groups in terms of unitary representations, and his first spectacular use of the notion was for showing that lattices in appropriate semi-simple groups are finitely generated [10]. As further natural examples were discovered, it was realized that the property makes sense for Hausdorff topological groups which need not be locally compact; see [3] and [18].

The property was also generalized from groups to pairs $H \subset G$ consisting of a closed subgroup H of a topological group G. This appears explicitely in Margulis' work on finitely-additive measures on Euclidean spaces [12]. But establishing this property for the pair $\mathbb{K}^2 \subset \mathbb{K}^2 \rtimes SL_2(\mathbb{K})$ is already the main step in Kazhdan's original proof that $SL_n(\mathbb{K})$ has Property (T) for a local field $\mathbb{K}$ and an integer $n \geq 3$, and Property (T) for pairs occurs also (without its name) in Margulis' explicit construction of concentrators (see Lemma 3.15 in [11]). More recently, in [15] and in [16], S. Popa proved rigidity results on type II_1 factors that need Property (T) for pairs of countable groups such as $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$. Thus Property (T) for pairs of topological groups has credentials to be viewed as a very basic notion.

It is standard that Property (T) for groups has several equivalent formulations (at least in the most important case of locally compact groups, possibly with some extra finiteness condition such as compact generation or σ-compactness). In 1995, T. Bates and G. Robertson stated some of these equivalences for pairs, with a claim that standard arguments apply. Since this last point is not strictly true, we offer here a complete proof of a slightly extended version of Theorem 1.1 in [2] [1].

We repeat below some definitions and state the main result. Further definitions and auxiliary results are given in Section 2; Section 3 contains a proof of the main theorem, and Section 4 is devoted to some consequences: we prove that if G is a σ-compact locally compact group, if $L \subset K \subset H$ are closed subgroups of G, then Property (T) for the pair $L \subset G$ is inherited by the pair $K \subset H$, provided that the homogeneous spaces G/H and K/L have invariant probability measures and L is normal in G. This was stated for discrete groups in Theorems 1.4 and 1.5 in [2], and proved for locally compact second countable groups in Proposition 3.1 of [9].

Definition 1.1. (1) Let G be a Hausdorff topological group and let $(\pi, \mathcal{H})$ be a unitary representation of G. For a subset $Q \neq \emptyset$ of G and a real number $\varepsilon > 0$, a vector $\xi \in \mathcal{H}$ is

*To appear in *L'Ens. Mathématique*, 2005

[1] A version of the present paper has been circulating since 1999 under the title "On relative property T"; it has been used on several occasions, among others by S. Popa in [15, 16]. For the present publication, part of the introduction, some prerequisites and Section 4 have been added.

(Q,ε)-*invariant* if

$$(\star) \qquad \sup_{g\in Q} \|\pi(g)\xi - \xi\| < \varepsilon\|\xi\|.$$

(Observe that if ξ satisfies $(\star)$ then $\xi \neq 0$.) Say that π *almost has invariant vectors* if, for any compact subset $Q \neq \emptyset$ of G and any $\varepsilon > 0$, there exist (Q,ε)-invariant vectors.
(2) Let moreover H be a closed subgroup of G. The pair $H \subset G$ has *Property (T)* if, for every unitary representation π of G which almost has invariant vectors, there exists a vector $\xi \neq 0$ which satisfies $\pi(h)\xi = \xi$ for every $h \in H$. In particular, the group G itself has Property (T) if the pair $G \subset G$ has Property (T).
(3) A *Kazhdan pair* (Q,ε) for the pair of groups $H \subset G$ consists of a compact subset $Q \neq \emptyset$ of G and a real number $\varepsilon > 0$ such that, whenever a unitary representation π of G has a vector ξ for which $(\star)$ holds, then π has a non-zero vector which is invariant by $\pi(H)$.

For a pair $H \subset G$ as above and a unitary representation π of G in $\mathcal{H}$, we denote by $\mathcal{H}^H$ the closed subspace of $\mathcal{H}$ of all $\pi(H)$-invariant vectors, and by ξ^H the orthogonal projection of $\xi \in \mathcal{H}$ on $\mathcal{H}^H$.

A complex-valued function φ on a Hausdorff topological group G is *of positive type* if it is continuous and if

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\varphi(g_i^{-1}g_j) \geq 0$$

for every integer $n \geq 1$, for all elements $g_1, \dots, g_n \in G$ and for all complex numbers $\alpha_1, \dots, \alpha_n$. If moreover $\varphi(1) = 1$, we say that φ is *normalized.*

A complex-valued function ψ on G is *conditionally of negative type* if it is continuous, if $\psi(g^{-1}) = \overline{\psi(g)}$ for every $g \in G$ and if

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\psi(g_i^{-1}g_j) \leq 0$$

for every integer $n \geq 1$, for all elements $g_1, \dots, g_n \in G$ and complex numbers $\alpha_1, \dots, \alpha_n$ that satisfy $\sum_{i=1}^{n} \alpha_i = 0$.

THEOREM 1.2. *Let G be a Hausdorff topological group and let H be a closed subgroup of G. Consider the following properties for the pair $H \subset G$.*

- *(a1) There exists a Kazhdan pair for the pair $H \subset G$.*
- *(a2) The pair $H \subset G$ has Property (T).*
- *(a3) There exists a non-empty compact subset Q of G and a positive number ε such that, for every unitary representation π of G for which $(\star)$ is true, then the restriction of π to H contains a non-zero finite-dimensional subrepresentation.*
- *(a4) [respectively (a4')] The restriction to H of every complex-valued [respectively real-valued] function on G which is conditionally of negative type is bounded.*
- *(b1) There exists a Kazhdan pair (Q,ε_0) for $H \subset G$ with the following property: for any $\delta > 0$ and for every unitary representation π of G which has a $(Q,\delta\varepsilon_0)$-invariant unit vector ξ, its orthogonal projection ξ^H on $\mathcal{H}^H$ satisfies $\|\xi - \xi^H\| \leq \delta$.*
- *(b2) For every $\delta > 0$, there exists a Kazhdan pair (Q,ε) for $H \subset G$ with the following property: for any unitary representation π of G which has a (Q,ε)-invariant unit vector ξ, its projection ξ^H satisfies $\|\xi - \xi^H\| \leq \delta$.*

(b3) If $(\varphi_j)_{j\in J}$ is a net of normalized functions of positive type on G which converges uniformly to 1 on compact sets, then

$$\lim_{j\in J} \sup_{h\in H} |\varphi_j(h) - 1| = \lim_{j\in J} \|\varphi_j|_H - 1\|_\infty = 0.$$

Then:

- *Properties (a1) and (a2) are equivalent with each other, they imply Property (a3) which implies Property (a4), and the latter is equivalent with Property (a4').*

- *Property (b1) implies Property (b2), the latter is equivalent to Property (b3), and both imply Property (a1). Moreover, if H is normal in G, Properties (b1), (b2) and (b3) are equivalent with each other.*

Finally, assume that the group G is locally compact and σ-compact. Then properties (a1), (a2), (a3), (a4), (b2) and (b3) are equivalent with each other.

REMARK. We have no example of a pair $H \subset G$ with Property (T), where G is σ-compact and locally compact and where H is not a normal subgroup of G, for which Property (b1) does not hold.

Acknowledgements. My warmest thanks go to Pierre de la Harpe for his valuable suggestions and comments and his considerable help in the presentation of this article, and to Bachir Bekka and Ghislain Jaudon for having detected a gap in the proof of Corollary 4.1 in a preliminary version.

2 Some prerequisites

We gather first some known facts on functions of positive type and on functions which are conditionally of negative type.

Let G be a Hausdorff topological group and let π be a unitary representation of G in $\mathcal{H}$. To every $\xi \in \mathcal{H}$, one associates a function of positive type $\varphi_{\pi,\xi}$ on G by $\varphi_{\pi,\xi}(g) = \langle \pi(g)\xi, \xi\rangle$ for all $g \in G$. Conversely, let φ be a function of positive type on G. The so-called GNS-construction shows that, for any such function φ, there exists a unique (up to unitary equivalence) triple $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ where π_φ is a unitary representation of G in $\mathcal{H}_\varphi$ and ξ_φ is a cyclic vector in $\mathcal{H}_\varphi$ which satisfies $\varphi(g) = \langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi\rangle$ for all $g \in G$.

A unitary representation ρ of G is *weakly contained* in a unitary representation π if every function $\varphi_{\rho,\xi}$ is a uniform limit on compact subsets of sums of functions $\varphi_{\pi,\eta}$. In particular, if $\rho = 1_G$ is the trivial representation, it is weakly contained in a representation π if and only if the latter almost has invariant vectors.

Consider next the set of functions which are conditionally of negative type on G. It is a cone which is closed in the topology of simple convergence. By a theorem of Schoenberg (1938), if $\psi : G \to \mathbb{R}$ is a continuous function such that $\psi(1) = 0$ and $\psi(g^{-1}) = \psi(g)$ for all $g \in G$, then ψ is conditionally of negative type if and only if the functions $e^{-t\psi}$ are of positive type for all real numbers $t > 0$. (See for example Theorem 5.16 in [5].) Moreover, there exist a real Hilbert space $\mathcal{H}_\psi$, an orthogonal representation π_ψ of G on $\mathcal{H}_\psi$ and a cocycle $b : G \to \mathcal{H}_\psi$ (*i.e.* $b(gh) = b(g) + \pi_\psi(g)b(h)$ $\forall g, h \in G$ and $b(1) = 0$) such that

$$\psi(g^{-1}h) = \|b(h) - b(g)\|^2 \quad \forall g, h \in G.$$

(See Proposition 5.14 in [5].) In particular, $\psi(g) \geq 0$ for all $g \in G$.

Let H be a closed subgroup of G and let ψ be a real-valued function on G which is conditionally of negative type. The following lemma, which extends Lemma 4.4 in [8], will be used in Section 3:

LEMMA 2.1. *Let G, H and ψ be as above. For $t > 0$, denote by $(\pi_t, \mathcal{H}_t, \xi_t)$ the cyclic representation of G associated with the function of positive type $e^{-t\psi}$. Then the restriction of ψ to H is bounded if and only if there exists some $t > 0$ such that the restriction of π_t to H contains a non-zero finite-dimensional subrepresentation.*

Proof. Assume that $\psi|_H$ is bounded and let $c > 0$ be such that $\psi(h) \leq c$ for every $h \in H$. Let $t > 0$ be arbitrary. We are going to prove that $\pi_t|_H$ contains the trivial one-dimensional representation, *i.e.* $\mathcal{H}_t$ contains a non-zero vector η which satisfies: $\pi_t(h)\eta = \eta$ for all $h \in H$. To do that, let C be closed convex hull of $\pi_t(H)\xi_t$; it is the closure in $\mathcal{H}_t$ of

$$\{\sum_{i=1}^n s_i\pi_t(h_i)\xi_t \ ; \ n \geq 1, \ s_1, \ldots, s_n \in [0, 1], \ \sum_i s_i = 1, \ h_1, \ldots, h_n \in H\}.$$

Then we claim that $\|\xi\| \geq e^{-tc/2}$ for all $\xi \in C$: indeed, if $s_1, \ldots, s_n \in [0, 1]$ are such that $\sum_i s_i = 1$ and if $h_1, \ldots, h_n \in H$, then

$$\begin{aligned} \|\sum_i s_i\pi_t(h_i)\xi_t\|^2 &= \sum_{i,j} s_is_j\langle\pi_t(h_i^{-1}h_j)\xi_t, \xi_t\rangle \\ &= \sum_{i,j} s_is_je^{-t\psi(h_i^{-1}h_j)} \\ &\geq \sum_{i,j} s_is_je^{-tc} = e^{-tc}. \end{aligned}$$

Let $\eta \in C$ be the element of minimal norm. Then $\pi_t(h)\eta = \eta$ for every $h \in H$ by uniqueness, and $\eta \neq 0$.

Before giving the proof of the converse, assume that $\psi|_H$ is unbounded and let $(h_n)_{n\geq 1} \subset H$ be a sequence such that $\psi(h_n) \to \infty$ as $n \to \infty$. Then we claim that, for every $t > 0$ and every unit vector $\xi \in \mathcal{H}_t \otimes \overline{\mathcal{H}_t}$, one has

$$(*) \qquad \lim_{n\to\infty} \|\pi_t(h_n) \otimes \overline{\pi_t}(h_n)\xi - \xi\| = \sqrt{2}.$$

Indeed, it suffices to prove $(*)$ for ξ in the linear span of $\{\pi_t(g)\xi_t \otimes \overline{\pi_t}(g')\xi_t \ ; \ g, g' \in G\}$. Thus, write

$$\xi = \sum_{i,j=1}^m a_{ij}\pi_t(g_i)\xi_t \otimes \overline{\pi_t}(g'_j)\xi_t$$

with $a_{ij} \in \mathbb{C}$ and $g_i, g'_j \in G$ for all $i, j = 1, \ldots, m$. As ξ has norm one, we have for every $n \geq 1$:

$$\begin{aligned} \|\pi_t(h_n) \otimes \overline{\pi_t}(h_n)\xi - \xi\|^2 &= 2 - 2\mathrm{Re}\left(\sum_{i,j,k,l=1}^m a_{ij}\overline{a_{kl}}\langle\pi_t(g_k^{-1}h_ng_i)\xi_t, \xi_t\rangle\langle\pi_t({g'}_l^{-1}h_n{g'}_j)\xi_t, \xi_t\rangle\right) \\ &= 2 - 2\mathrm{Re}\left(\sum_{i,j,k,l=1}^m a_{ij}\overline{a_{kl}}\exp(-t[\psi(g_k^{-1}h_ng_i) + \psi({g'}_l^{-1}h_ng'_j)])\right) \end{aligned}$$

because ψ is real-valued. Let $(\mathcal{H}_\psi, \pi_\psi, b)$ be a triple as above: $\mathcal{H}_\psi$ is a real Hilbert space, π_ψ is an orthogonal representation of G on $\mathcal{H}_\psi$ and b is a cocycle such that

$$\psi(g^{-1}h) = \|b(h) - b(g)\|^2 \quad \forall g, h \in G.$$

Thus, $\|b(h_n)\| \to \infty$ as $n \to \infty$, which implies that

$$\begin{aligned}
\psi(g_k^{-1}h_n g_i) &= \|b(h_n g_i) - b(g_k)\|^2 \\
&= \|b(h_n) + \pi_\psi(h_n)b(g_i) - b(g_k)\|^2 \\
&\geq (\|b(h_n)\| - \|b(g_k) - \pi_\psi(h_n)b(g_i)\|)^2
\end{aligned}$$

tends to ∞ as $n \to \infty$ for all $1 \leq i, k \leq m$ and similarly for $\psi({g'}_l^{-1}h_n g'_j)$ for all $1 \leq j, l \leq m$. This proves $(*)$.

If the restriction of π_t to H contains a non-zero finite-dimensional subrepresentation for some $t > 0$, as is well known, this means that $(\pi_t \otimes \overline{\pi}_t)|_H$ contains the trivial representation of H, and $(*)$ implies that ψ is bounded. □

The following two lemmas are certainly well known to some readers, but we include their proofs for convenience.

LEMMA 2.2. *Let H be a group, let π be a unitary representation of H in a Hilbert space $\mathcal{H}$ and let δ be a positive number. If there exists a unit vector $\xi \in \mathcal{H}$ such that $\|\pi(h)\xi - \xi\| \leq \delta$ for all $h \in H$, then $\|\xi^H - \xi\| \leq \delta$.*

Proof. As in the proof of Lemma 2.1, let C be the closed convex hull of $\pi(H)\xi$. By assumption, one has $\|\zeta - \xi\| \leq \delta$ for all $\zeta \in C$. Since ξ^H is the element of minimal norm in C, we get the conclusion. □

Assume now that G is a locally compact group and let H be a closed subgroup of G. Following Definition 6.1.4 in [4], we say that H is **co-Følner** in G if there exists a G-invariant state on $L^\infty(G/H)$ (equivalently, if the homogeneous space G/H is amenable in the sense of Eymard [6]). It amounts to say that the quasi-regular representation $\lambda_{G/H}$ of G on $L^2(G/H)$ almost has invariant vectors. Observe that if there exists a G-invariant probability measure on G/H then H is co-Følner in G. The next lemma will be needed in Section 4. LEMMA 2.3. *Let $H \subset G$ be as above and assume that H is co-Følner in G. If π is a unitary representation of H which almost has invariant vectors, then its induced representation $\sigma = Ind_H^G(\pi)$ almost has invariant vectors, too.*

Proof. If π almost has invariant vectors, then the trivial representation 1_H of H is weakly contained in π, and, by continuity of inducing, $\lambda_{G/H} = \mathrm{Ind}_H^G(1_H)$ is weakly contained in σ. Since H is co-Følner in G, $\lambda_{G/H}$ almost has invariant vectors, and so does σ. □

3 Proof of Theorem 1.2

To begin with, Properties (a4) and (a4') are equivalent because of the following standard fact. Let $\psi : G \to \mathbb{C}$ be a function which is conditionally of negative type. Then $\mathrm{Re}(\psi(g) - \psi(1)) \geq 0$ for all $g \in G$ and the real-valued function ψ_0 defined by $\psi_0(g) = \mathrm{Re}\{(\psi(g) - \psi(1))^{1/2}\}$ is also conditionally of negative type. Moreover, ψ and ψ_0 are together bounded or not on G. (See Corollary 5.19 in [5].)

The implication (a1) $\Longrightarrow$ (a2) is straightforward.

(a2) implies (a1). Let I be the set of all pairs (Q, ε) where Q is a non empty compact subset of G and $\varepsilon > 0$. If (a1) does not hold, then for every $i := (Q, \varepsilon) \in I$, one can find a unitary representation π_i in $\mathcal{H}_i$ of G and a unit vector $\xi_i \in \mathcal{H}_i$ such that

$$\sup_{g \in Q} \|\pi_i(g)\xi_i - \xi_i\| < \varepsilon,$$

but such that $\mathcal{H}_i^H = \{0\}$. Set $\pi = \bigoplus_{i \in I} \pi_i$. It almost has invariant vectors, hence, by (a2), it has some non-zero vector which is invariant by $\pi(H)$, and this implies that $\mathcal{H}_j^H \neq \{0\}$ for some j, which contradicts the choice of π_j.

(a1) implies (a3) obviously.

(a3) implies (a4') follows readily from Lemma 2.1.

(b1) implies (b2) is trivial because (b2) is a special case of (b1).

(b2) implies (b3). Let $(\varphi_j)_{j\in J}$ be a net as in (b3) and denote by $(\pi_j, \mathcal{H}_j, \xi_j)$ the GNS-representation of φ_j for all $j \in J$. Fix a positive number δ, and let (Q, ε) be a Kazhdan pair associated to $\delta/2$. There exists $j_\delta \in J$ such that

$$\sup_{g\in Q} |\varphi_j(g) - 1| < \frac{\varepsilon^2}{2}$$

for all $j \geq j_\delta$. Hence $\|\pi_j(g)\xi_j - \xi_j\| = \sqrt{2\text{Re}(1 - \varphi_j(g))} < \varepsilon$ for all $g \in Q$ and all $j \geq j_\delta$. Thus, one has, by (b2), $\|\xi_j^H - \xi_j\| \leq \delta/2$ for all $j \geq j_\delta$. We get for all $h \in H$ and all $j \geq j_\delta$:

$$\begin{aligned} |\varphi_j(h) - 1| &= |\langle \pi_j(h)\xi_j - \xi_j, \xi_j\rangle| \\ &\leq \|\pi_j(h)\xi_j - \xi_j\| \\ &\leq \|\pi_j(h)\xi_j - \xi_j^H\| + \|\xi_j^H - \xi_j\| \\ &\leq 2\|\xi_j^H - \xi_j\| \leq \delta. \end{aligned}$$

(b3) implies (b2). Let again I be the set of all pairs (Q, ε) as in the proof of *(a2) implies (a1)*. I is a net when gifted with the following partial ordering: $(Q, \varepsilon) \preceq (Q', \varepsilon')$ if and only if $Q \subset Q'$ and $\varepsilon \geq \varepsilon'$. If (b2) does not hold, there exists $\delta > 0$ such that for every pair $i = (Q, \varepsilon) \in I$ one can find a representation π_i on $\mathcal{H}_i$, a unit vector $\xi_i \in \mathcal{H}_i$ and $h_i \in H$ such that

$$\sup_{g\in Q} \|\pi_i(g)\xi_i - \xi_i\| < \varepsilon$$

but

$$\|\pi_i(h_i)\xi_i - \xi_i\| \geq \delta$$

by Lemma 2.2. Put $\varphi_i(g) = \langle \pi_i(g)\xi_i, \xi_i\rangle$. Then $\varphi_i \to 1$ uniformly on compact sets hence

$$\sup_{h\in H} |\varphi_i(h) - 1| \to 0.$$

But one has $\|\pi_i(h_i)\xi_i - \xi_i\| \geq \delta$ for every i, which implies that $\text{Re}(1 - \varphi_i(h_i)) \geq \frac{\delta^2}{2}$. This gives a contradiction.

(b2) implies (a1). This is obvious.

Assume now that H is normal in G and that Property (b2) holds. Choose $\delta > 0$ and let (Q, ε) be as in (b2). Set $\varepsilon_0 = \varepsilon/2$. If $(\pi, \mathcal{H})$ is a unitary representation of G which has a unit vector ξ that satisfies

$$\sup_{g\in Q} \|\pi(g)\xi - \xi\| \leq \frac{\delta\varepsilon}{2}$$

then we just need to prove that

$$\sup_{h\in H} \|\pi(h)\xi - \xi\| \leq \delta.$$

To do that, set $M(\xi) = \sup_{g\in Q} \|\pi(g)\xi - \xi\|$, and let $\mathcal{H}_1$ be the orthogonal complement of $\mathcal{H}^H$ in $\mathcal{H}$. Since H is normal in G, both subspaces are G-invariant. Write $\xi = \eta + \zeta$ with

$\eta \in \mathcal{H}^H$ and $\zeta \in \mathcal{H}_1$. If $\zeta = 0$, then $\xi = \eta = \xi^H$, and we are done. Assume that $\zeta \neq 0$. Then one has for every $g \in Q$:

$$\begin{aligned}\|\pi(g)\frac{\zeta}{\|\zeta\|} - \frac{\zeta}{\|\zeta\|}\| &= \frac{1}{\|\zeta\|}\|\pi(g)\zeta - \zeta\| \\ &\leq \frac{1}{\|\zeta\|}\|\pi(g)\xi - \xi\| \leq \frac{M(\xi)}{\|\zeta\|}.\end{aligned}$$

But necessarily $M(\xi) \geq \varepsilon\|\zeta\|$ because the restriction of π to $\mathcal{H}_1$ has no non-zero H-invariant vector. Thus $\|\xi - \eta\| = \|\zeta\| \leq \varepsilon^{-1}M(\xi)$, and we get for all $h \in H$:

$$\|\pi(h)\xi - \xi\| = \|\pi(h)(\xi - \eta) - (\xi - \eta)\| \leq 2\|\xi - \eta\| \leq \frac{2}{\varepsilon}M(\xi) \leq \delta.$$

Hence, when H is a normal subgroup of G, Properties (b1), (b2) and (b3) are equivalent with each other.

Finally, let us assume that G is a locally compact, σ-compact group. In order to complete the proof of Theorem 1.2, it remains to prove that Property (a4) implies Property (b3). Indeed, we already know that

$$(a1) \iff (a2) \implies (a3) \implies (a4)$$

and that

$$(b2) \iff (b3) \implies (a1).$$

(a4) implies (b3). (Adapted from [1].) Suppose that there exists a sequence $(\varphi_n)_{n\geq 1}$ of functions which are normalized and of positive type on G such that $\varphi_n \to 1$ uniformly on compact sets, but with

$$\liminf_{n\to\infty} \|\varphi_n|_H - 1\|_\infty = 2\varepsilon > 0.$$

Then there exist $(h_n) \subset H$ and integers $k_n \geq n$ such that

$$|\varphi_{k_n}(h_n) - 1| \geq \varepsilon$$

for all $n \geq 1$. Let $(Q_n)_{n\geq 1}$ be an increasing sequence of compact subsets of G such that $G = \bigcup_n Q_n$. Taking a subsequence if necessary, we assume that

$$\sup_{g\in Q_n} |\varphi_n(g) - 1| \leq 4^{-n}$$

and

$$|\varphi_n(h_n) - 1| \geq \varepsilon$$

for every n. Using the inequality:

$$|\varphi(s) - \varphi(t)|^2 \leq 2(1 - \mathrm{Re}\varphi(s^{-1}t))$$

for all $s, t \in G$ and for every normalized function of positive type φ on G, we get:

$$\frac{\varepsilon^2}{2} \leq \mathrm{Re}(1 - \varphi_n(h_n))$$

for every n. Then set

$$\psi(g) = \sum_{n=1}^{\infty} 2^n \,\mathrm{Re}(1 - \varphi_n(g)),$$

which is a function conditionally of negative type on G. As

$$\psi(h_n) \geq 2^{n-1}\varepsilon^2$$

for every n, we see that ψ is unbounded on H.

The proof is now complete. □

4 Consequences

Let G be a Hausdorff topological group and let H, K, L be closed subgroups of G such that $L \subset K \subset H \subset G$. Observe that, if the pair $K \subset H$ has Property (T), it follows from the definition that the pair $L \subset G$ has also Property (T). Theorem 1.2 has the following consequence, which extends Theorem 1.5 in [2] and Proposition 3.1 in [9].

Corollary 4.1. *The notation being as above, assume moreover that G is a second countable, locally compact group and that the pair $L \subset G$ has Property (T).*

(1) If there exists a K-invariant probability measure on K/L, then the pair $K \subset G$ has Property (T).

(2) If H is co-Følner in G and if L is a normal subgroup of G, then the pair $L \subset H$ has Property (T).

Particular case. *Let H be a locally compact, second countable group and let K be a closed subgroup of H. Assume that there exists a H-invariant probability measure on H/K. Then the three following properties are equivalent:*

(i) the group H has Property (T),

(ii) the group K has Property (T),

(iii) the pair $K \subset H$ has Property (T).

The equivalence between (i) and (ii) is already in [10]. In particular, it holds for a lattice in a locally compact group. It is the last step of Kazhdan's argument for showing that lattices in appropriate semi-simple Lie groups (in particular in affine algebraic groups over local fields which are simple of split rank at least two) are finitely-generated.

Proof of Corollary 4.1. (1) Let μ be a K-invariant probability measure on K/L and denote by p the canonical projection from K onto K/L. Let $(\pi, \mathcal{H})$ be a unitary representation of G which almost has invariant vectors. We are going to prove that $\mathcal{H}$ contains non-zero invariant vectors for $\pi(K)$. As the pair $L \subset G$ satisfies Property (b2), given some $0 < \delta < 1/16$, there exists a Kazhdan pair (Q, ε) as in (b2), and we assume that $\varepsilon \leq \delta$. Choose a compact set $C \subset K$ such that, if $X = p(C)$, then $\mu(K/L \setminus X) < \delta$, and choose a unit vector $\xi \in \mathcal{H}$ which satisfies

$$\sup_{g \in Q \cup C} \|\pi(g)\xi - \xi\| \leq \varepsilon.$$

Then there exists a unit vector $\eta \in \mathcal{H}^L$ such that $\|\eta - \xi\| \leq 2\delta$. In particular, we get for all $k \in C$:

$$(*) \qquad \|\pi(k)\eta - \xi\| \leq \|\pi(k)\eta - \pi(k)\xi\| + \|\pi(k)\xi - \xi\| \leq 3\delta.$$

Let us define then $\eta' : K/L \to \mathcal{H}$ by $\eta'(kL) = \pi(k)\eta$ for all $k \in K$. Finally, set

$$\eta_1 = \int_{K/L} \eta'(x) d\mu(x) \in \mathcal{H}.$$

As μ is a K-invariant measure, one has $\pi(k)\eta_1 = \eta_1$ for all $k \in K$, and we are left to prove that $\eta_1 \neq 0$. But, by the inequalities $(*)$, we have $\|\eta'(x) - \xi\| \leq 3\delta$ for all $x \in X$, and this implies that

$$\| \int_X \eta'(x) d\mu(x) - \mu(X)\xi \| \leq 3\delta$$

hence

$$\| \int_X \eta'(x) d\mu(x) \| \geq \mu(X) - 3\delta \geq 1 - 4\delta > \frac{3}{4}.$$

Finally, the inequality $\|\eta_1 - \int_X \eta'(x)d\mu(x)\| \leq \delta$ implies that $\|\eta_1\| \geq \frac{3}{4} - \delta > 0$.

(2) Let π be a unitary representation of H which almost has invariant vectors. By Lemma 2.3, its induced representation $\sigma = \mathrm{Ind}_H^G(\pi)$ almost has invariant vectors, too. Thus its restriction to L has an invariant unit vector ξ. We realize σ as in [7], p. 348, and we keep S. Gaal's notations: choose a quasi-invariant probability measure μ on G/H and let $\mathcal{K} = \mathcal{K}(\mu)$ be the Hilbert space of (classes of) measurable functions $\eta : G \to \mathcal{H}$ such that, for all $h \in H$, $\eta(gh) = \pi(h^{-1})\eta(g)$ dg-a.e., and $\int_{G/H} \|\eta(x)\|^2 d\mu(x) < \infty$. Finally, denote by $\lambda(g, x)$ the Radon-Nikodym derivative given by the action of G on G/H. Hence $\xi : G \to \mathcal{H}$ is a Borel function such that:

(a) for every $h \in H$, one has $\xi(gh) = \pi(h^{-1})\xi(g)$ dg-a.e.,

(b) $\displaystyle\int_{G/H} \|\xi(x)\|^2 d\mu(x) = 1$,

(c) for every $l \in L$, $\xi(lg)\sqrt{\lambda(l, \dot{g})} = \xi(g)$ dg-a.e..

We claim that G/H has an L-invariant probability measure: indeed, set

$$\nu(B) = \int_{G/H} \chi_B(x) \|\xi(x)\|^2 d\mu(x)$$

for every Borel subset B of G/H. Then Property (c) above shows that ν is L-invariant. As ν is a regular measure, its support S is a closed, L-invariant subset of G/H. Let $Y \subset G$ be the preimage of S under the canonical projection of G onto G/H. Y is a closed subset of G, it has positive Haar measure, it is left L-invariant and right H-invariant. Replacing μ by $\mu' = \mu(S)\nu|_S + (1 - \mu(S))\mu|_{G/H \setminus S}$, we get a quasi-invariant probability measure on G/H such that the Radon-Nikodym derivative $\lambda'(h, s) = 1$ for $(l, s) \in L \times S$. Realizing σ on the Hilbert space $\mathcal{K}(\mu')$, the non-zero vector $\xi' = \xi\sqrt{\frac{d\mu}{d\mu'}}$ restricted to Y satisfies:

(a') for every $h \in H$, one has $\xi'(yh) = \pi(h^{-1})\xi'(y)$ for almost all $y \in Y$,

(b') $\displaystyle\int_{G/H} \|\xi'(x)\|^2 d\mu(x) = 1$,

(c') for every $l \in L$, $\xi'(ly) = \xi'(y)$ for almost all $y \in Y$.

Consider next the following right actions of $L \times H$ on Y and on $\mathcal{H}$:

$$y \cdot (l, h) = l^{-1}yh \qquad \text{and} \qquad \eta \cdot (l, h) = \pi(h^{-1})\eta.$$

Then, by Proposition B.5 of [19], there is an $L \times H$-invariant conull Borel subset $Y_0 \subset Y$ and a Borel $L \times H$-map $\xi'' : Y_0 \to \mathcal{H}$ such that $\xi'' = \xi'$ a.e.. In particular, one has for all $y \in Y_0$ and all $(l, h) \in L \times H$:

$$\xi''(l^{-1}yh) = \pi(h^{-1})\xi''(y).$$

Thus, choose $y \in Y_0$ such that $\xi''(y) \neq 0$. Then, since L is normal in G and contained in H, one has for every $l \in L$:

$$\pi(l)\xi''(y) = \xi'(yl^{-1}) = \xi''(yl^{-1}y^{-1}y) = \xi''(y)$$

which means that $\xi''(y)$ is a non-zero L-invariant vector. □

EXAMPLES. (1) The existence of a K-invariant probability measure on K/L cannot be replaced by the amenability of K/L: indeed, let H be a countable, infinite, amenable group, let L be any finite subgroup of H and let K be an infinite subgroup of H which contains L. Then the pair $L \subset H$ obviously has Property (T), but the pair $K \subset H$ does not: the regular representation of H almost has invariant vectors, and its restriction to K has no non-zero invariant vectors.
(2) We give an example where Corollary 4.1 applies: Set $\Gamma_0 = SL_2(\mathbb{Z})$, let $\Gamma_1 = [\Gamma_0, \Gamma_0]$ be the commutator subgroup of Γ_0 and set $\Gamma_2 = [\Gamma_1, \Gamma_1]$. It is proved in [6] that Γ_2 is co-Følner in $SL_2(\mathbb{R})$, but that the homogeneous space $SL_2(\mathbb{R})/\Gamma_2$ has no $SL_2(\mathbb{R})$-invariant probability measure. As the pair $\mathbb{R}^2 \subset \mathbb{R}^2 \rtimes SL_2(\mathbb{R})$ has Property (T), the pairs $\mathbb{R}^2 \subset \mathbb{R}^2 \rtimes \Gamma_2$ and $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma_2$ have also Property (T).

REMARKS. (1) We don't know whether the second statement of Corollary 4.1 remains true if L is not normal in G.
(2) Recently, R. Nicoara, S. Popa and R. Sasyk found a characterization of Property (T) for pairs of countable groups in terms of projective representations which is similar to our condition (b2): see Theorem 3.1 in [13].
(3) Let H be a closed subgroup of a topological group G. Say that G has *Property (T) relative to H* if any unitary representation π of G which almost has invariant vectors and which has non-zero invariant vectors by H has also non-zero vectors invariant by G. (In particular, Property (T) for the group G itself means that G has Property (T) relative to $\{1\}$.) This property plays its role in Popa's articles [14] and [17], but *it should not* be confused with Property (T) for the pair $H \subset G$.

References

[1] AKEMANN, C. A. and M. E. WALTER. Unbounded negative definite functions. *Canad. J. Math. 33* (1981), 862–871.

[2] BATES, T. and A. G. ROBERTSON. Positive definite functions and relative Property (T) for subgroups of discrete groups. *Bull. Austral. Math. Soc. 52* (1995), 31–39.

[3] BEKKA, B. Kazhdan's Property (T) for the unitary group of a separable Hilbert space. *Geom. Funct. Anal. 13* (2003), 509–520.

[4] CHERIX, P.-A., M. COWLING, P. JOLISSAINT, P. JULG and A. VALETTE. *Groups with the Haagerup property (Gromov's a-T-menability).* Birkhäuser (Basel), 2001.

[5] DE LA HARPE, P. and A. VALETTE. *La propriété (T) de Kazhdan pour les groupes localement compacts.* Astérisque, 1989.

[6] EYMARD, P. Moyennes invariantes et représentations unitaires. Lecture Notes in Mathematics no. 300. Springer (New-York), 1972.

[7] Gaal, S. *Linear analysis and representation theory.* Springer (New-York), 1973.

[8] Jolissaint, P. Property T for discrete groups in terms of their regular representation. *Math. Ann. 297* (1993), 539–551.

[9] Jolissaint, P. Borel cocycles, approximation properties and relative Property T. *Ergod. Th. & Dynam. Syst. 20* (2000), 483–499.

[10] Kazhdan, D. Connection of the dual space of a group with the structure of its closed subgroups. *Funct. Anal. Appl. 1* (1967), 63–65.

[11] Margulis, G. A. Explicit constructions of concentrators. *Problems Inform. Transmission 9* (1973), 325-332.

[12] Margulis, G. A. Finitely-additive invariant measures on Euclidean spaces. *Ergod. Th. & Dynam. Syst. 2* (1982), 383–396.

[13] Nicoara, R., S. Popa and R. Sasyk. *Some remarks on irrational rotation HT factors.* Preprint, 2004.

[14] Popa, S. *Correspondences.* Unpublished, 1986.

[15] Popa, S. *On a class of type II_1 factors with Betti numbers invariants.* MSRI preprint, 2001-005.

[16] Popa, S. *Some rigidity results for non-commutative Bernoulli shifts.* MSRI preprint, 2001-024, revised version math. OA/0209130.

[17] Popa, S. Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T. *Documenta Math. 4* (1999), 665–744.

[18] Shalom, Y. Bounded generation and Kazhdan's Property (T). *Publ. Math. I.H.E.S. 90* (1999), 145–168.

[19] Zimmer, R. *Ergodic Theory and Semisimple Groups.* Birkhäuser (Basel), 1984.

Institut de Mathémathiques,
Université de Neuchâtel,
Emile-Argand 11
CH-2000 Neuchâtel, Switzerland
paul.jolissaint@unine.ch