Spaces with measured walls, Haagerup property and property (T)

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Abstract

We introduce the notion of space with measured walls, generalizing the concept of space with walls due to Haglund and Paulin [HP98]. We observe that if a locally compact group G acts properly on a space with measured walls, than G has the Haagerup property. We conjecture that the converse holds, and we prove this conjecture for the following classes of groups: discrete groups with the Haagerup property, closed subgroups of $SO(n,1)$, groups acting properly on real trees, $SL_2(K)$ where K is a global field, and amenable groups.

1 Introduction

Let G be a locally compact, second countable group. We say that G has *property (T)* if every continuous, affine, isometric action on a Hilbert space has a fixed point (see [dlHV89]); and that G has the *Haagerup property* if G admits a continuous, affine, isometric action on some Hilbert space, which is proper (see [CCJ+01]).

The notion of space with walls was introduced by Haglund and Paulin [HP98] to provide a unified framework between trees, planar tessellations by $2n$-gons, cubical $CAT(0)$-complexes, etc...

Definition 1 *Let X be a set and $\mathcal{W}$ be a set of partitions of X into 2 classes; call these partitions* **walls**. *The pair $(X, \mathcal{W})$ is a* **space with walls** *if, for every pair x, y of distinct points in X, the number $w(x, y)$ of walls separating x from y, is finite.*

It was observed by Haglund, Paulin and the third author that the automorphism group of a space with walls has a natural affine, isometric action on a Hilbert space, to the effect that: if a locally compact group G admits a proper action on a space with walls, then it has the Haagerup property; if a locally compact group G with property (T) acts on a space with walls, then all orbits are bounded (see [CCJ$^+$01], Cor. 7.4.2).

Since spaces with walls are discrete spaces, we cannot hope for a converse of these implications. So let us consider the following generalization.

Definition 2 *Let X be a set, $\mathcal{W}$ a set of walls on X, $\mathcal{B}$ a σ-algebra of sets in $\mathcal{W}$, and μ a measure on $\mathcal{B}$. The 4-tuple $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a* **space with measured walls** *if, for every pair x, y of distinct points in X, the set $\omega(x, y)$ of walls separating x from y belongs to $\mathcal{B}$, and $w(x, y) =: \mu(\omega(x, y))$ is finite.*

If, in Definition 2, the measure μ is counting measure, we recover Definition 1.

If $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a space with measured walls, it is easy to see that the function

$$X \times X \to \mathbb{R}^+ : (x, y) \mapsto w(x, y)$$

is a pseudo-distance (or *écart*) on X: we call it the *wall metric*. It makes sense of bounded subsets in X, or of proper actions of a group on X. As we explain in §2, the automorphism group of a space with measured walls has a natural affine, isometric action on a Hilbert space, hence:

Proposition 1 *Let G be a locally compact group.*

(1) If G admits a proper action on a space with measured walls, then G has the Haagerup property.

(2) If G has property (T), then every action of G on a space with measured walls, has bounded orbits.

Concerning the Haagerup property, our main result is the following partial converse of Proposition 1(1). We conjecture that this converse should hold in general.

Theorem 1 *The following groups with the Haagerup property admit a proper action on a space with measured walls:*

(1) *discrete groups with the Haagerup property;*

(2) *closed subgroups of $SO(n,1)$, the group of isometries of real hyperbolic space of dimension $n \geq 1$;*

(3) *groups acting properly isometrically on real trees;*

(4) *the groups $SL_2(K)$, $PGL_2(K)$, $SL_2(\mathbb{A}_K)$ and $PGL_2(\mathbb{A}_K)$, where K is a global field and $\mathbb{A}_K$ is its ring of adèles;*

(5) *amenable groups.*

This result is proved in sections §§2-5. Theorem 1(1) is basically a rephrasing of Proposition 7.5.1 in [CCJ$^+$01], and depends crucially on results of Robertson and Steger [RS98], as we explain in §2. Theorem 1(2) was actually our motivating example: it is a re-interpretation of a result of Robertson [Rob98], for which we offer a slightly different proof in §3.

Concerning property (T), our main results are as follows.

Theorem 2 *Let Γ be a countable group. The group Γ has property (T) if and only if every action of Γ on a space with measured walls, has bounded orbits.*

This will be proven in §2. If H, Γ are countable groups, we recall that the *wreath product* $H \wr \Gamma$ is the semi-direct product of $\bigoplus_\Gamma H$, the direct sum of copies of H indexed by Γ, with Γ acting by shifting indices. Using spaces with measured walls, we prove in §5 [1]:

Theorem 3 *Let H, Γ be non-trivial countable groups. The following are equivalent:*

(i) *$H \wr \Gamma$ has property (T);*

(ii) *H has property (T) and Γ is finite.*

In §6, we give an application of spaces with measured walls, in the form of a modest contribution to the relation between ordered groups and property (T). The final section, §7, collects some open questions suggested by our study.

[1]That result has been obtained independently by M. Neuhauser [Neu], with another proof.

2 An exercise in unification

Let X be a set. Recall the following concepts:

- A function $\Psi : X \times X \to \mathbb{R}^+$ is a **conditionally negative definite kernel** if $\Psi(x,x) = 0$ and $\Psi(x,y) = \Psi(y,x)$ for every $x, y \in X$, and for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i = 0$, one has:
$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \Psi(x_i, x_j) \leq 0.$$
- A function $\Psi : X \times X \to \mathbb{R}^+$ is a **measure definite kernel** in the sense of Robertson and Steger [RS98] if there exists a measure space $(\Omega, \mathcal{B}, \mu)$ and a family of sets $S_x \in \mathcal{B}$ $(x \in X)$ such that
$$\Psi(x,y) = \mu(S_x \triangle S_y)$$
for every $x, y \in X$.

Every measure definite kernel is conditionally negative definite ([RS98], Prop. 1.1); if Ψ is a conditionally negative kernel and X is *countable*, then $\sqrt{\Psi}$ is a measure definite kernel ([RS98], Prop.1.2).

The next observation gives the relation between measure definite kernels and spaces with measured walls.

Proposition 2 *(1) Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a space with measured walls. The function $X \times X \to \mathbb{R}^+ : (x,y) \mapsto w(x,y)$ is a measure definite kernel.*

(2) Let Ψ be a measure definite kernel on a countable set X. Then X carries a structure of space with measured walls $(X, \mathcal{W}, \mathcal{B}, \mu)$ such that $w(x,y) = \Psi(x,y)$ for every $x, y \in X$. Moreover, if X is a principal homogeneous space over a group Γ, and Ψ is Γ-invariant, then Γ acts by automorphisms on $(X, \mathcal{W}, \mathcal{B}, \mu)$.

Proof: (1) Define a *half-space* in X as a class of the partition of X defined by some wall in $\mathcal{W}$. Let Ω denote the set of half-spaces. Let $p : \Omega \to \mathcal{W}$ be the map which, to a half-space, associates the wall it belongs to. This map p is a "double cover" in the sense that all fibers of p have cardinality 2. Define a σ-algebra $\mathcal{A}$ on Ω by
$$\mathcal{A} = \{A \subset \Omega : p(A) \in \mathcal{B}\},$$

and a measure ν on $\mathcal{A}$ by

$$\nu(A) = \frac{1}{2} \int_{p(A)} card(p^{-1}(w) \cap A) \, d\mu(w).$$

By construction $p_* \nu = \mu$. For $x \in X$, let S_x be the set of half-spaces through x. For distinct $x, y \in X$, we have $p^{-1}(\omega(x,y)) = S_x \triangle S_y$, so that

$$w(x,y) = p_* \nu(\omega(x,y)) = \nu(S_x \triangle S_y),$$

showing that the kernel w on X is measure definite.

(2) Let Ψ be a measure definite kernel on the countable set X. Let $\Omega = \{0,1\}^X \backslash \{(0,0,0,\ldots),(1,1,1,\ldots)\}$ be the set of non-empty, non-full subsets of X. For $x \in X$, let S_x be the set of subsets of X through x. By Proposition 1.2 in [RS98], since X is countable, there exists a Borel measure ν on Ω such that $\nu(S_x \triangle S_y) = \Psi(x,y)$ for every $x, y \in X$.

Define then a wall of X as a partition $\{B, B^c\}$, where $B \in \Omega$. The set $\mathcal{W}$ of walls identifies with the quotient of Ω by the fixed point free involution $B \mapsto B^c$ Denote by $\mathcal{B}$ the direct image of the Borel σ-algebra on Ω, and by μ the direct image of ν on $\mathcal{W}$. Then $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a space with measured walls, such that $\Psi(x,y) = w(x,y)$ for every $x, y \in X$. To prove the second statement, observe that Γ clearly acts on $\mathcal{W}$ and $\mathcal{B}$; moreover the measure μ is Γ-invariant, by the proof of Theorem 2.1 in [RS98]. $\square$

Proof of Proposition 1, Theorem 1(1), and Theorem 2:

Let G be a second countable, locally compact group. We recall that a function $\psi : G \to \mathbb{R}^+$ is conditionally negative definite if the function $G \times G \to \mathbb{R}^+ : (g,h) \mapsto \psi(g^{-1}h)$ is a conditionally negative definite kernel. Moreover:

- the group G has property (T) if and only if every continuous, conditionally negative definite function on G is bounded (see [dlHV89]);
- the group G has the Haagerup property if and only if there exists a continuous, proper, conditionally negative definite function on G (see [CCJ+01]).

The combination of these characterizations, Proposition 2, and the Robertson-Steger results mentioned above, proves Proposition 1, Theorem 1(1) and Theorem 2. $\square$

Remarks: Proposition 1 encompasses a number of known results on spaces with walls.

1) A tree is a space with walls (every edge divides the set of vertices into two subsets, hence can be seen as a wall), and every action of a group with property (T) on a tree has bounded orbits, hence fixes either some vertex or some edge (Watatani [Wat81]).

2) Any $CAT(0)$ cubical complex carries a structure of space with walls (by a result of Sageev [Sag95], every bisecting hyperplane divides the complex into two components), and every action of a group with property (T) on a CAT(0) cubical complex has bounded orbits, hence has a fixed point (Niblo-Reeves [NR97]).

   This example is actually general: as was proven implicitly by Sageev [Sag95] and explicitly by Chatterji-Niblo [CN] and independently Nica [Nic], to any space with walls $(X, \mathcal{W})$ is associated in a natural way a $CAT(0)$ cubical complex $Cub(X, \mathcal{W})$, in such a way that any group G acting (resp. acting properly) on $(X, \mathcal{W})$ also acts (resp. acts properly) on $Cub(X, \mathcal{W})$.

3 Real hyperbolic spaces

In this short section, we explain how a result of Robertson [Rob98] fits into our framework.

Let $X = \mathbb{H}^n(\mathbb{R})$ be real n-dimensional hyperbolic space, and let $G = Isom\, X$ be its isometry group (so that G is locally isomorphic to $SO(n, 1)$).

A *hyperplane* in X is a totally geodesic hypersurface. Every hyperplane defines in two ways a *wall*, i.e. a closed half-space and its complementary open half-space. The group G acts transitively on walls; since the stabilizer H of a given wall is unimodular, the space $\mathcal{W}$ of walls carries a G-invariant measure μ. For every $x, y \in X$, the set $\omega(x, y)$ of walls separating x from y is relatively compact in $\mathcal{M} = G/H$, so that $w(x, y) < \infty$ (see [Rob98] for details).

Denote by $d(x, y)$ the hyperbolic distance between x and y in X. The following Crofton formula was proved by Robertson [Rob98]. We give a proof that is somewhat different - and, to our taste, simpler - as suggested by E. Ghys.

Proposition 3 *There exists a constant $\lambda > 0$ (only depending of normalizations of G-invariant measures) such that, for every $x, y \in X$:*

$$w(x, y) = \lambda d(x, y).$$

Proof: Note first that the function w is clearly measurable on $X \times X$. Since X is 2-point homogeneous, i.e. G acts transitively on pairs on equidistant points, $w(x, y)$ only depends on $d(x, y)$, i.e. there is a measurable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$w(x, y) = \phi(d(x, y))$$

for every $x, y \in X$. Fix $r_1, r_2 \geq 0$ and choose three collinear points $x, y, z \in X$ such that $d(x, y) = r_1$, $d(y, z) = r_2$ and $d(x, z) = r_1 + r_2$. Since the set of walls whose hyperplane goes through y, has measure zero, we have

$$w(x, y) + w(y, z) = w(x, z).$$

This means that ϕ satisfies the functional equation

$$\phi(r_1) + \phi(r_2) = \phi(r_1 + r_2),$$

i.e. ϕ is additive. Since ϕ is measurable (and non-zero), there exists $\lambda > 0$ such that $\phi(r) = \lambda r$ for every $r \geq 0$. This concludes the proof. $\square$

Proof of Theorem 1(2): Since any closed subgroup of G acts properly on X, the result is clear from the previous Proposition. $\square$

4 Real trees

We recall that an *arc* in a metric space X, is a subset homeomorphic to any compact interval of $\mathbb{R}$, while a *segment* in X is a subset isometric to some interval of $\mathbb{R}$.

Definition 3 *A* **real tree** *is a metric space (X, d) such that any two distinct points $x, y \in X$ belong to a unique arc $[x, y]$, which moreover is a closed segment.*

In this section, X will always denote a real tree. We first describe how a real tree carries a canonical structure of space with measured walls.

Fix $x \in X$. Among all open segments $]x, y[$ (with $y \in X$), define a relation $\sim$ by setting:

$$]x, y[\sim]x, z[\Longleftrightarrow]x, y[\cap]x, z[\neq \emptyset.$$

This is clearly an equivalence relation. A class of this relation is a *germ of segments* at x.

We will denote by TX the set of pairs (x, σ) where $x \in X$ and σ is a germ of segments at x. Let $\Pi : TX \to X : (x, \sigma) \mapsto x$ be the canonical projection. For x, y distinct points in X, we set:

$$I_{xy} = \{(z, \sigma) \in TX : z \in [x, y[,]z, y[\in \sigma\}.$$

We call such sets *local sections* (since $\Pi|_{I_{xy}}$ is a bijection from I_{xy} to $[x, y[$). The *length* of I_{xy} is $d(x, y)$. Denote by $\mathcal{B}$ the σ-algebra on TX generated by all local sections.

Lemma 1 *There exists a measure μ on $\mathcal{B}$ such that $\mu(I_{xy}) = d(x, y)$ for every $x, y \in X$.*

Proof: The proof is quite similar to the proof of existence of "Lebesgue measure" on X, defined on the σ-algebra generated by segments in X (see Proposition 3 in [Val90]). We consider the ring $\mathcal{R}$ of subsets of TX generated by local sections: this is the set of disjoint unions of local sections. For $B \in \mathcal{R}$, let us choose a finite decomposition of B into local sections, and define $\mu(B)$ as the sum of the lengths of these local sections. This definition does not depend on the chosen decomposition. If $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements in $\mathcal{R}$, with $\bigcup_{n\geq 1} B_n \in \mathcal{R}$, properties of Lebesgue measure on $\mathbb{R}$ do imply:

$$\mu(\bigcup_{n\geq 1} B_n) = \sum_{n\geq 1} \mu(B_n).$$

By the theorem on extension of measures (see §13.A in [Hal50]), we may extend μ uniquely to a σ-additive measure on the σ-algebra $\mathcal{B}$. $\square$

To every $(x, \sigma) \in TX$, we associate a wall in X, defined as follows:

$$W_{(x,\sigma)} = \{y \in X :]x, y[\in \sigma\};$$

$$W'_{(x,\sigma)} = X - W_{(x,\sigma)}.$$

Proposition 4 *Let (X,d) be a real tree. The set TX endowed with the measure μ defines on X a structure of space with measured walls such that $w(x,y) = 2d(x,y)$ for every $x, y \in X$.*

Proof: Let x, y be two distinct points in X. For $(z,\sigma) \in \omega(x,y)$, we have either $x \in W_{(z,\sigma)}$ and $y \in W'_{(z,\sigma)}$, or the converse. Let us consider the first case. From the structure of triangles in X, we must have $z \in [y, x[$, and actually the set of (z,σ)'s such that $x \in W_{(z,\sigma)}$ and $y \in W'_{(z,\sigma)}$ coincides with local section I_{yx}. So $w(x,y) = \mu(I_{yx}) + \mu(I_{xy}) = 2d(x,y)$. □

Proof of Theorem 1(3): Obvious in view of the previous Proposition. □

5 Products of spaces with measured walls

Definition 4 *Let $(X_1, \mathcal{W}_1, \mathcal{B}_1, \mu_1)$, $(X_2, \mathcal{W}_2, \mathcal{B}_2, \mu_2)$ be spaces with measured walls. Let $p_i : X_1 \times X_2 \to X_i$ $(i = 1,2)$ be the projection on the i-th factor. Set $\mathcal{W} = p_1^{-1}(\mathcal{W}_1) \amalg p_2^{-1}(\mathcal{W}_2)$, let $\mathcal{B}$ be the σ-algebra on $\mathcal{W} \simeq \mathcal{W}_1 \amalg \mathcal{W}_2$ by $\mathcal{B}_1 \amalg \mathcal{B}_2$, and let μ be the unique measure on $\mathcal{W}$ such that $\mu_{|\mathcal{B}_i} = \mu_i$ $(i = 1,2)$. Then $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a space with walls, called the* **product** *of the two original ones.*

It is possible to define the product of a countable family of spaces with measured walls, but only for *pointed* spaces, i.e. a base-point has been chosen in each factor.

Definition 5 *Let $(X_n, \mathcal{W}_n, \mathcal{B}_n, \mu_n, x_n^0)_{n \geq 1}$ be a sequence of pointed spaces with measured walls. Set*

$$X = \{(x_n)_{n \geq 1} \in \prod_{n=1}^{\infty} X_n : \sum_{n=1}^{\infty} w_n(x_n, x_n^0) < \infty\}.$$

Denote by $p_n : X \to X_n$ the projection onto the n-th factor. Set

$$\mathcal{W} = \coprod_{n=1}^{\infty} p_n^{-1}(\mathcal{W}_n) \simeq \coprod_{n=1}^{\infty} \mathcal{W}_n,$$

let $\mathcal{B}$ be the σ-algebra on $\mathcal{W}$ generated by $\coprod_{n=1}^{\infty} \mathcal{B}_n$, and let μ be the unique measure on $\mathcal{W}$ such that $\mu_{|\mathcal{B}_n} = \mu_n$ for every n. Then $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a space with measured walls, called the **product** *of the $(X_n, \mathcal{W}_n, \mathcal{B}_n, \mu_n, x_n^0)$'s.*

Note that, in that situation, for $x = (x_n)_{n\geq 1}, y = (y_n)_{n\geq 1}$ two points in X, one has:

$$w(x, y) = \sum_{n=1}^{\infty} w_n(x_n, y_n),$$

since the series on the right hand side converges.

Proof of Theorem 1(4): Since $SL_2(K)$ (resp. $PGL_2(K)$) is a lattice in $SL_2(\mathbb{A}_K)$ (resp. $PGL_2(\mathbb{A}_K)$), it is enough to consider the latter group. Since the natural homomorphism $SL_2(\mathbb{A}_K) \to PGL_2(\mathbb{A}_K)$ has compact kernel, we actually reduce to $PGL_2(\mathbb{A}_K)$.

Let us denote by P the set of places of K, and by P_f the subset of finite places. For $v \in P$, we denote by K_v the completion of K at v.

Assume $v \in P_f$; we denote by $\mathcal{O}_v$ the valuation ring of K_v, and by π_v a uniformizer of $\mathcal{O}_v$ (i.e. π_v generates the unique maximal ideal in $\mathcal{O}_v$). Let X_v be the tree of $PGL_2(K_v)$, with vertex set $PGL_2(K_v)/PGL_2(\mathcal{O}_v)$ (see [Ser77]). As base-point in X_v, we choose the vertex x_v^0 with stabilizer $PGL_2(\mathcal{O}_v)$. Let ν_v be counting measure on the edges of X_v; we view X_v as a space with measured walls, with the measure

$$\mu_v = \frac{\nu_v}{|\pi_v|_v}$$

on edges.

Assume now $v \notin P_f$; there are two cases. If v is real, i.e. $K_v = \mathbb{R}$, we let $PGL_2(\mathbb{R})$ act on real hyperbolic plane $X_v = \mathbb{H}^2(\mathbb{R}) = PGL_2(\mathbb{R})/PO(2)$; as base-point, we choose the point x_v^0 with stabilizer $PO(2)$. If v is complex, i.e. $K_v = \mathbb{C}$, we let $PGL_2(\mathbb{C})$ act on real hyperbolic 3-space $X_v = \mathbb{H}^3(\mathbb{R}) = PGL_2(\mathbb{C})/PU(2)$; as base-point, we choose the point x_v^0 with stabilizer $PU(2)$. In both cases we view X_v as a space with measured walls, as in §3.

Since $PGL_2(\mathbb{A}_K)$ is the restricted product of the $PGL_2(K_v)$'s with respect to the $PGL_2(\mathcal{O}_v)$'s, we see that $PGL_2(\mathbb{A}_K)$ acts on the direct product X of the $(X_v, \mathcal{W}_v, \mathcal{B}_v, \mu_v)$'s, for $v \in P$. It remains to see that the action of $PGL_2(\mathbb{A}_K)$ on X is proper. For this, it is enough to prove that, with $x^0 = (x_v^0)_{v\in P}$, the function $g \mapsto w(gx^0, x^0)$ is proper on $PGL_2(\mathbb{A}_K)$, i.e. we must show that, for $R \geq 0$ the set

$$W_R = \{g \in PGL_2(\mathbb{A}_K) : w(gx^0, x^0) \leq R\}$$

is compact. But, for an element $g = (g_v)_{v \in P}$ in $PGL_2(\mathbb{A}_K)$, we have:

$$\begin{aligned} w(gx^0, x^0) &= \sum_{v \in P_f} \frac{d_v(g_v x_v^0, x_v^0)}{|\pi_v|_v} + \sum_{v \in P - P_f} w_v(g_v x_v^0, x_v^0) \\ &= \sum_{v \in P_f} \frac{d_v(g_v x_v^0, x_v^0)}{|\pi_v|_v} + \sum_{v \in P - P_f} \lambda_v d_v(g_v x_v^0, x_v^0) \end{aligned}$$

by Proposition 3.

If v is an infinite place (there are finitely many such places), the condition $d_v(g_v x_v^0, x_v^0) \leq \frac{R}{\lambda_v}$ defines a compact subset C_v in $PGL_2(K_v)$. To treat finite places, we first observe that the set

$$S = \{v \in P_f : |\pi_v|_v^{-1} \leq R\}$$

is finite. For $v \notin S$, the condition $d_v(g_v x_v^0, x_v^0) \leq R|\pi_v|_v$ is equivalent to $g_v \in PGL_2(\mathcal{O}_v)$. For $v \in S$, the same condition defines a compact subset C_v in $PGL_2(K_v)$. So W_R is contained in the compact subset

$$\prod_{v \in S \cup (P - P_f)} C_v \times \prod_{v \in P_f - S} PGL_2(\mathcal{O}_v)$$

of $PGL_2(\mathbb{A}_K)$. This concludes the proof. □

Remarks:

1) The previous result generalizes Example 6.1.2 in [CCJ+01], by showing that the function

$$g \mapsto = \sum_{v \in P_f} \frac{d_v(g_v x_v^0, x_v^0)}{|\pi_v|_v} + \sum_{v \in P - P_f} d_v(g_v x_v^0, x_v^0)$$

is measure-definite on $PGL_2(\mathbb{A}_K)$.

2) Let $\mathbb{D}$ be a quaternion algebra over K. Denote by $\mathbb{D}^\times$ its multiplicative group, and by $Z(\mathbb{D}^\times)$ the centre of $\mathbb{D}^\times$. Set $G = \mathbb{D}^\times / Z(\mathbb{D}^\times)$, viewed as an algebraic group over K. Then the groups $G(K)$ and $G(\mathbb{A}_K)$ admit a proper action on a space with measured walls: the proof is similar to the previous one, by noticing that, for every place v of K, the group $G(K_v)$ is isomorphic to $PGL_2(K_v)$ if $\mathbb{D}$ splits at v, while $G(K_v)$ is compact otherwise (see [Vig80]).

We now turn to other applications of products of spaces with measured walls.

Example 1 *Let G be a locally compact group, with left Haar measure m. Fix a compact subset A of G containing the identity, and a positive number $\lambda > 0$. We will define a space with measured walls $X_{A,\lambda} = (G, \mathcal{W}, \mathcal{B}, \mu)$, where $\mathcal{W} = \{\{gA^{-1}, (gA^{-1})^c\} : g \in G\}$. Set $K_A = \{g \in G : gA^{-1} = A^{-1}\}$, a compact subgroup in G. Then $\mathcal{W}$ identifies with G/K_A, so we denote by $\mathcal{B}$ the σ-algebra of Borel subsets in G/K_A, and by $\overline{m}$ the G-invariant measure on G/K_A coming from m. For $B \in \mathcal{B}$, we set $\mu(B) = \lambda\overline{m}(B)$. In this way we construct $X_{A,\lambda}$, which is pointed by the identity of G. Note that, for $x, y \in G$, one has: $w(x, y) = \lambda m(xA \triangle yA)$, i.e. the Haar measure of the symmetric difference between xA and yA, up to a factor λ.*

Note that, if G is discrete (so that m is counting measure), the space $X_{A,1}$ is a space with walls in the sense of Haglund-Paulin.

The space with measured walls $X_{A,\lambda}$ is very innocuous, since the associated conditionally negative definite kernel is bounded. However, we shall see that, suitably varying A and λ, and taking products, we can get something more interesting. Actually, this is completely similar to the proof in [BCV95] that amenable groups have the Haagerup property: by packing up suitably chosen coboundaries, one obtains a proper cocycle.

Proof of Theorem 1(5): Let G be a σ-compact amenable group. Let $(K_n)_{n\geq 1}$ be an increasing, exhaustive sequence of compact subsets in G. Amenability allows us to find a Følner sequence, i.e. a sequence $(A_n)_{n\geq 1}$ of compact subsets such that

$$\frac{m(gA_n \triangle A_n)}{m(A_n)} < 2^{-n}$$

for every $g \in K_n$ and $n \geq 1$. Set $\lambda_n = \frac{n}{m(A_n)}$, and form the product X of the spaces X_{A_n,λ_n}. Observe then that G acts on X by the diagonal action: indeed, set first $x_0 = (1, 1, 1, \ldots) \in X$ and notice that, for $g \in G$:

$$w(gx_0, x_0) = \sum_{n=1}^{\infty} \frac{n \cdot m(gA_n \triangle A_n)}{m(A_n)}$$

which converges since, for n large enough: $g \in K_n$, so the tail of the sequence is dominated by the convergent sequence $\sum_{n\geq n_0}^{\infty} n2^{-n}$; now, for $x = (x_n)_{n\geq 1}$

in X, and $g \in G$, we have, by the triangle inequality and G-invariance of m:

$$w(gx, x_0) \leq w(gx, gx_0) + w(gx_0, x_0) = w(x, x_0) + w(gx_0, x_0) < \infty.$$

It remains to show that G acts properly on X, i.e. that the function $g \mapsto w(gx_0, x_0)$ is proper on G. So fix $R > 0$, we need to show that $\{g \in G : w(gx_0, x_0) \leq R\}$ is compact in G. Choose $n \geq R$: we get $\frac{m(gA_n \triangle A_n)}{m(A_n)} \leq \frac{R}{n} \leq 1$, which yields $m(gA_n \cap A_n) \geq m(A_n)$. The conclusion follows from the fact that the function $g \mapsto m(gA_n \cap A_n)$ on G is continuous with compact support.
□

Proof of Theorem 3: Assume first that Γ is finite. Then, by standard results on property (T) (see [dlHV89]), we have that H has property (T) if and only if $\bigoplus_\Gamma H$ has property (T), if and only if $H \wr \Gamma$ has property (T). This proves $(ii) \Rightarrow (i)$, and also $(i) \Rightarrow (ii)$ in case Γ is finite. To conclude the proof of $(i) \Rightarrow (ii)$, we assume that Γ is infinite and show that $H \wr \Gamma$ admits an action with unbounded orbits on a space with measured walls: by Proposition 1, $H \wr \Gamma$ does not have property (T).

Consider a family, indexed by Γ, of copies of the space with walls $X_{\{1\},1}$ associated with H. Form the product X of this family. The group $\bigoplus_\Gamma H$ acts componentwise on X, while Γ acts by shifting indices; these two actions combine into an action of $H \wr \Gamma$, and it remains to prove that this action has unbounded orbits. Fix a non-trivial element $h \in H$. For a finite subset $A \subset \Gamma$, define an element $g_A \in \bigoplus_\Gamma H$ by

$$(g_A)_\gamma = \begin{cases} h & \text{if } \gamma \in A \\ 1 & \text{if } \gamma \notin A \end{cases}$$

Then $w(g_A x_0, x_0) = \sum_{\gamma \in A} w_\gamma(h, 1) = 2|A|$, showing that the function $g \mapsto w(gx_0, x_0)$ is not bounded on $H \wr \Gamma$. □

Note that the space X appearing in the previous proof is a space with walls in the sense of Haglund-Paulin. So we get, as an immediate consequence:

Corollary 1 *Let H be a non-trivial countable group with property (T). For a countable group Γ, the following are equivalent:*

(i) $H \wr \Gamma$ has property (T);

(ii) every action of $H \wr \Gamma$ on a space with walls has bounded orbits;

(iii) Γ is finite.

$\square$

As pointed out to us by M. Neuhauser, another immediate consequence of the proof of Theorem 3 is the following:

Corollary 2 *Let H, Γ be non-trivial countable groups. If the pair $(H \wr \Gamma, \bigoplus_\Gamma H)$ has the relative property (T), then Γ is finite.*

$\square$

6 Ordered groups

Definition 6 *Let G be a locally compact group. We say that G is* **ordered** *if G is endowed with a left-invariant, total ordering $<$ such that $P = \{g \in G : g \geq 1\}$ is a Borel subset in G.*

If G is discrete, we recover the standard concept of an ordered group.

If G is an ordered, locally compact group, and $g < h$ in G, the *segment* $[g, h[$ is $[g, h[= \{x \in G : g \leq x < h\}$. Note that $[g, h[= gP \cap hP^c$, so that segments are Borel subsets.

Lemma 2 *Let G be a discrete, non-trivial, ordered group. Assume that segments in G are finite. Then G is isomorphic to $\mathbb{Z}$ as an ordered group.*

Proof: Let $g \in G$ be such that $g > 1$. Then $[1, g[$ is a chain $1 < g_1 < g_2 < \ldots < g_n$. Let us show that G is generated by g_1. For this, we prove by induction on k that, if $h > 1$ and $[1, h[$ has k elements, then $h = g_1^k$. This is clear if $k = 1$, by comparing h and g_1. Assume then $k > 1$, and observe that, since $g_1 < h$:

$$k - 1 = |[g_1, h[| = |[1, g_1^{-1}h[|.$$

So by induction hypothesis: $g_1^{-1}h = g_1^{k-1}$, which concludes the proof. $\square$

The following is a small contribution to the question: can an ordered group have property (T)?

Proposition 5 *Let G be a non-trivial, ordered, locally compact group. Assume that every segment in G has finite Haar measure. Then G does not have property (T).*

Proof: Denote by m the Haar measure on G. We define a space with measured walls $(G, \mathcal{W}, \mathcal{B}, m)$. Set $\mathcal{W} = \{\{gP, gP^c\} : g \in G\}$. Since the map $G \to \mathcal{W} : g \mapsto \{gP, gP^c\}$ is a bijection, we may endow $\mathcal{W}$ with the Borel σ-algebra $\mathcal{B}$ of G, and with the measure m. Note that, for $g < h$ in G, one has $w(g,h) = m([g,h[)$. Set $\psi(g) = w(g,1)$ for $g \in G$.

Assume by contradiction that G has property (T). By the previous lemma, we may assume G non-discrete. Since ψ is a conditionally negative definite function on G, and G has property (T), there exists $M \geq 0$ such that $\psi(g) \leq M$ for every $g \in G$. For $g > 1$ and $n \in \mathbb{N} - \{0\}$, we have

$$\psi(g^n) = m([1, g^n[) = m([1, g[\cup[g, g^2[\cup \ldots \cup [g^{n-1}, g^n[)$$

$$= \sum_{i=1}^{n} m(g^{i-1}[1, g[) = n\psi(g) \leq M.$$

Since this holds for every n, we deduce $\psi(g) = 0$ for $g \in P$. Since $\psi(g) = \psi(g^{-1})$, we see that ψ vanishes identically. This means that, for every $g \in G$:

$$m(gP \triangle P) = 0.$$

By the ergodicity of the action of G by left multiplications on itself, we deduce that either $m(P) = 0$ or $m(P^c) = 0$. On the other hand $P^c \cup \{1\} = P^{-1}$. Since G is unimodular and non-discrete, we get $m(P^c) = m(P)$, so in both cases we get $m(G) = 0$, which is clearly a contradiction. $\square$

Unfortunately, the only examples we know of groups satisfying the assumptions of Proposition 5, are $\mathbb{Z}$ and $\mathbb{R}$. However, Proposition 5 has a somewhat unexpected consequence.

Corollary 3 *A locally compact group containing a non-trivial compact subgroup, cannot be ordered.*

Proof: Let K be a non-trivial compact group: on the one hand, K has property (T); on the other hand, any Borel subset of K has finite Haar measure. By Proposition 5, K cannot be ordered.

The corollary now follows by observing that the property of being ordered, is inherited by closed subgroups. □

Remark: It is obvious that a group containing a non-trivial element with finite order, cannot be ordered. However, there exists torsion-free compact groups: think of the additive group $\mathbb{Z}_p$ of the ring of p-adic integers (p a prime). Note that $\mathbb{Z}_p$ can be ordered as a discrete group, since it is isomorphic to an additive subgroup of $\mathbb{R}$ (reason: as a $\mathbb{Q}$-vector space, the p-adic field $\mathbb{Q}_p$ is isomorphic to $\mathbb{R}$). The previous corollary shows that the positive cone cannot be a Borel subset in $\mathbb{Z}_p$.

Notice that the proof of Proposition 5 appeals to two ingredients only:

- the function $g \mapsto m([1, g[)$ is bounded on P;
- m is finitely additive and G-invariant.

So the same proof also gives:

Corollary 4 *Let G be an amenable, ordered, locally compact group. For every invariant mean ν defined on the Borel subsets of G, and every $g > 1$ in G, one has: $\nu[1, g[= 0$.*

□

Remark: A countable group G can be ordered if and only if G can be embedded as a subgroup of the group $Homeo^+(\mathbb{R})$ of orientation-preserving homeomorphisms of the real line (see [Wit94]). It is known (see [Ghy01]) that any connected, locally compact subgroup of $Homeo^+(\mathbb{R})$ surjects either onto $\mathbb{R}$ or onto $\widetilde{SL_2(\mathbb{R})}$, so it does not have property (T).

7 Open questions

1. *Is it true that every locally compact group with the Haagerup property, admits a proper action on a space with measured walls?* In view of Theorem 1, we conjecture that the answer is yes. The test case here seems to be $SU(n, 1)$: as observed already in [Rob98], the proof of Theorem 1 for $SO(n, 1)$ breaks down right from the start for $SU(n, 1)$, for the reason that hyperplanes do not separate in complex hyperbolic space $\mathbb{H}^n(\mathbb{C})$. It is an open problem to adapt the proof for $SO(n, 1)$

to $SU(n,1)$, by finding a family of real hypersurfaces which disconnect $\mathbb{H}^n(\mathbb{C})$.

2. *Are there discrete groups with the Haagerup property which cannot act properly on a space with walls?* This question was already discussed by Chatterji and Niblo [CN], who proved the following: let Γ be a group containing an amenable subgroup with super-polynomial growth; if Γ acts properly on a space with walls, then there are arbitrarily large families of walls in the space which cross pairwise (two walls cross if the four involved half-spaces pairwise meet); equivalently, Γ cannot act properly on a finite-dimensional $CAT(0)$ cubical complex.

3. *Is there an intrinsic characterization of measure definite kernels coming from a space with walls? From a space with measured walls?* It is known that every measure-definite kernel is *hypermetric* [2]: this was proved in [Kel70], Theorem 3.1. The following short, elegant proof of this fact is due to G. Robertson (private communication): let Ψ be a measure-definite kernel on X; let $(\Omega, \mathcal{B}, \mu)$ be a measure space such that $\Psi(x,y) = \mu(\Sigma_x \Delta \Sigma_y)$. Denote by χ_x the characteristic function of Σ_x; then
$$\Psi(x,y) = \int_\Omega |\chi_x(\omega) - \chi_y(\omega)| \, d\mu(\omega).$$
So for $x_1, \dots, x_n \in X$, $t_1, \dots, t_n \in \mathbb{Z}$ with $\sum_{i=1}^n t_i = 1$, we have:
$$\sum_{i=1}^n \sum_{j=1}^n t_i t_j \Psi(x_i, x_j) \;=\; \int_\Omega \sum_{i=1}^n \sum_{j=1}^n t_i t_j |\chi_{x_i}(\omega) - \chi_{x_j}(\omega)| \, d\mu(\omega).$$
It is enough to show that the integrand is non-negative. Fix $\omega \in \Omega$, and define $\delta : X \to \{0,1\} : x \mapsto \chi_x(\omega)$; then the integrand factors out as
$$\sum_{i=1}^n \sum_{j=1}^n t_i t_j |\delta(x_i) - \delta(x_j)| \;=\; 2P_0 P_1$$
where $P_k = \sum_{i:\delta(x_i)=k} t_i$ for $k \in \{0,1\}$. Observe now that P_0, P_1 are integers summing up to 1, so that $2P_0P_1 \leq 0$.

   Examples of hypermetric kernels which are *not* measure-definite can be found in [DGL95].

[2] A kernel Ψ on a set X is hypermetric if for every finite set $\{x_1, \dots, x_n\}$ in X and integers $t_1, \dots, t_n$ such that $\sum_{i=1}^n t_i = 1$, one has $\sum_{i=1}^n \sum_{j=1}^n t_i t_j \Psi(x_i, x_j) \leq 0$.

Quite related to this question is the following: give an intrinsic characterization of those conditionally negative definite functions on a group G which come from an action of G on a space with walls (resp. space with measured walls). For groups acting on trees, such an abstract characterization was obtained by Chiswell [Chi76].

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