Curvature, Harnack's Inequality, and a Spectral Characterization of Nilmanifolds

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Abstract. For closed n–dimensional Riemannian manifolds M with almost positive Ricci curvature, the Laplacian on one–forms is known to admit at most n small eigenvalues. If there are n small eigenvalues, or if M is orientable and has $n-1$ small eigenvalues, then M is diffeomorphic to a nilmanifold, and the metric is almost left invariant. We show that our results are optimal for $n \geq 4$.

1. Introduction

A classical theorem of Bochner states that the first real Betti number of a closed n–dimensional Riemannian manifold M with positive semi–definite Ricci curvature tensor Ric satisfies the inequality $b_1(M) \leq n$, with equality only if M is isometric to a flat torus. This result is a consequence of Weitzenböck's formula for the Hodge–de Rham–Laplacian $\Delta = d\delta + \delta d$ on one–forms α,

$$\Delta\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha^\sharp, \cdot). \tag{1.1}$$

The formula implies that all harmonic one–forms on M are parallel with respect to the Levi–Civita connection of the metric. Since the space of parallel one–forms has dimension at most n, Bochner's Betti number estimate is a consequence of the Hodge theorem on harmonic forms. And if $b_1(M) = n$, then the Albanese map obtained by integrating an L^2-orthonormal basis of the space of harmonic forms yields an isometry of M with its Albanese torus.

Bochner's inequality for $b_1(M)$ has been extended by Gallot ([Ga1] Cor. 3.2) and Gromov ([Gr] p. 73) to include manifolds whose Ricci tensor and diameter satisfy

$$\mathrm{Ric}\ \mathrm{diam}^2(M) \geq -\varepsilon(n) \tag{1.2}$$

for suitably small positive $\varepsilon(n)$ depending only on n. The case of equality was settled only recently by Cheeger and Colding ([CCo] p. 459) to the effect that (1.2) and $b_1(M) = n$ still imply that M is diffeomorphic to the torus. But it appears to be unknown whether a diffeomorphism is given by the Albanese map.

Gallot and Meyer ([GaM]) extended Bochner's theorem in a different direction by giving an explicit bound for the number of small eigenvalues of the Laplacian, instead of only the multiplicity $b_1(M)$ of the zero eigenvalue. Consider a compact connected Riemannian manifold (M, g) without boundary, of dimension n and diameter $\mathrm{diam}(M, g) \leq d$. Let

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots$$

1991 Mathematics Subject Classification 58G25, 53C20.

Key words: Nilmanifolds, Laplacian, Harnack inequality.

This work was supported by a Swiss NSF grant 20–50579.97 and by the Forschungsinstitut für Mathematik at ETH Zürich.

denote the spectrum of Δ on one–forms, with each eigenvalue repeated according to its multiplicity. Assuming a Ricci curvature bound $\mathrm{Ric}\, d^2 \geq -\varepsilon$ for a real number ε, the result of Gallot and Meyer ([GaM] p. 574, see also [Ga3]), when specialized to λ_{n+1}, states that

$$\lambda_{n+1} d^2 \geq \frac{\lambda^* d^2}{8(n+1)^2} - \varepsilon. \tag{1.3}$$

Here λ^* denotes the smallest positive eigenvalue of the Laplacian on functions. Lower bounds for λ^* in terms of Ricci curvature and diameter were obtained by Li and Yau. In particular, Theorem 10 in [LiY] states that

$$\lambda^* d^2 + \max\{0, \varepsilon\} \geq \pi^2/4. \tag{1.4}$$

Combined with (1.3), this yields a positive lower bound on λ_{n+1}, provided ε is not too positive. So Δ can have at most n small eigenvalues.

In [PeS] and [CGR], the authors considered what happens when Δ actually does have n small eigenvalues. Petersen and Sprouse showed in [PeS] that, under an additional bound on the curvature tensor R, M has to be diffeomorphic to an *infra–nilmanifold*, i.e. a quotient of a nilpotent Lie group by a discrete group of isometries of some left invariant Riemannian metric. Compact infra–nilmanifolds admit finite covering spaces that are nilmanifolds. In [CGR1], under bounds on R and its covariant derivative ∇R, M itself was shown to be diffeomorphic to a nilmanifold. In this paper, we generalize and sharpen both results.

For $m \in M$, let $\underline{\mathrm{Ric}}(m)$ denote the lowest eigenvalue of the Ricci tensor $\mathrm{Ric}(m)$, considered as a symmetric operator on $T_m M$. For a function $f : M \to \mathbf{R}$, we denote by $f^-(m) = \max\{0, -f(m)\}$ its negative part. Our main result is the following

Theorem 1.1. *For every dimension n and real number $p > n$, there is a positive constant $\varepsilon(n,p)$ such that the following is true. Suppose (M^n, g) satisfies* $\mathrm{diam}(M,g) \leq d$ *and*

$$\|\underline{\mathrm{Ric}}^-\|_{p/2}\, d^2 \leq \varepsilon(n,p)\big(1 + \|R\|_{q/2}\, d^2\big)^{-\beta(n,p)} \tag{1.5}$$

$$\lambda_n d^2 \leq \varepsilon(n,p)\big(1 + \|R\|_{q/2}\, d^2\big)^{-\beta(n,p)} \tag{1.6}$$

with $q = \max\{p, 4\}$ *and*

$$\beta(n,p) = \frac{(p+n)(q-2)}{p-n}.$$

Then M is diffeomorphic to a nilmanifold. If instead of (1.6) *we have*

$$\lambda_{n-1} d^2 \leq \varepsilon(n,p)\big(1 + \|R\|_{q/2}\, d^2\big)^{-\beta(n,p)}, \tag{1.7}$$

then M is diffeomorphic to a nilmanifold or to a non–orientable infra–nilmanifold. In either case, the metric g is close to a left invariant metric g_0 for the nilpotent structure in the sense that, for $k = n$ or $n-1$ respectively,

$$\|g - g_0\|_\infty \leq \delta\big(\|\underline{\mathrm{Ric}}^-\|_{p/2}\, d^2 + \lambda_k d^2,\ n\big)$$

for some function δ such that $\delta(t,n) \to 0$ as t tends to zero.

The volume–normalized $L^{p/2}$–norms used in this statement are defined in section 2, and $\|\cdot\|_\infty$ denotes the maximum norm on tensor fields. We note that little would be lost if we normalized the diameter bound to $d = 1$. In the form given, our inequalities are scaling invariant. A *nilmanifold* is a quotient $\Gamma\backslash G$ of a nilpotent Lie group G by a discrete subgroup Γ of G. The left invariant Riemannian metrics on G descend to such quotients, and are called left invariant metrics on $\Gamma\backslash G$ by abuse of language.

Remarks 1.2. **(i)** It is well known, and will be explained in section 5, that every compact nilmanifold admits left invariant metrics with $\|R\|_\infty d^2$ and $\lambda_n d^2$ arbitrary small, so that a converse of our result holds. This is not true for general infra–nilmanifolds.

(ii) Instead of pointwise curvature bounds, only integral norms of the curvature enter our hypotheses. This is a less restrictive assumption, as Gallot [Ga2] and Yang [Yan] have shown that there are sequences of Riemannian manifolds of diameter one with uniform bounds on $\|\underline{\mathrm{Ric}}^-\|_{p/2}$ or $\|R\|_{q/2}$, that do not admit Riemannian metrics with diameter one and uniform pointwise lower bounds on Ric, or upper bounds on $\|R\|_\infty$, respectively. For such metrics, the somewhat delicate smoothing argument required in (and briefly indicated on page 84 of) [PeS] may not apply.

(iii) In addition to a Ricci curvature bound, our assumptions involve the norm $\|R\|_{q/2}$ of the full curvature tensor. Counterexamples given in section 4 show that the result does not hold without a bound on R, even if we assume smallness of the Ricci tensor in the L^∞–norm.

(iv) Compact nilmanifolds with first Betti number equal to n are known to be tori. In fact, a result of Nomizu (see [Ra] p. 123) states that the real cohomology of a compact nilmanifold $\Gamma\backslash G$ is isomorphic to the cohomology of the Lie algebra of G; so $b_1(\Gamma\backslash G) = n$ implies that G is abelian. As a consequence, our result includes a weak form of the theorem of Cheeger and Colding, weak as it involves a bound on R. Under the present assumptions, our proof shows that the Albanese map is a diffeomorphism. The statement on λ_{n-1} extends and sharpens a theorem of Yamaguchi (see [Yam]).

(v) The proof of Theorem 1.1 remains of use when there are only $k < n-1$ small eigenvalues, in the sense that hypothesis (1.6) in Theorem 1.1 is replaced by

$$\lambda_k d^2 \le \varepsilon(n,p)\bigl(1 + \|R\|_{q/2}\, d^2\bigr)^{-\beta(n,p)}.$$

In that case, the immediate conclusion is that TM contains a trivial subbundle of rank k. In particular, we obtain a lower bound on the first eigenvalue λ_1 of the Laplacian on one–forms in terms of dimension, diameter and $\|R\|_{q/2}$ for manifolds with $\mathrm{Ric} \ge 0$ and non–vanishing Euler characteristic.

For manifolds with non–negative Ricci tensor, (1.3) and (1.4) imply that

$$\lambda_{n+1} d^2 \ge \frac{\pi^2}{32(n+1)^2}.$$

If, in addition, (1.7) holds, then g is a metric of non–negative Ricci curvature on a compact infra–nilmanifold. The splitting theorem of Cheeger and Gromoll (see [ChG] p. 126) implies that such metrics are flat, and we obtain the following

Corollary 1.3. *Suppose* (M, g) *satisfies* $\mathrm{Ric} \geq 0$. *If*

$$\lambda_{n-1} d^2 \leq \varepsilon(n,p)\big(1 + \|R\|_{q/2}\, d^2\big)^{-\beta(n,p)}$$

then (M, g) *is isometric to a euclidean space form; and if*

$$\lambda_n d^2 \leq \varepsilon(n,p)\big(1 + \|R\|_{q/2}\, d^2\big)^{-\beta(n,p)}$$

then (M, g) *is isometric to a flat torus.*

Our proof of Theorem 1.1 is based on a Harnack inequality for generalized Schrödinger operators. This inequality allows us to employ a result from [Gh1] characterizing nilmanifolds instead of the L^2–pinching theorem of [MiR] used in [PeS]. Unlike the gradient estimates given in [PeS] and [CGR], this method does not require bounds on the covariant derivative ∇R of the curvature tensor.

The article is structured as follows. Section 2 contains the Harnack and regularity estimates required for the proof of Theorem 1.1. The proof proper is given in the following section. In section 4, we describe examples explaining our curvature assumptions, while section 5 is devoted to spectral properties of infra–nilmanifolds.

We refer to the monograph [Sa] for notation and general background in Riemannian geometry, and to [Bé] for an introduction to Bochner methods and eigenvalue estimates.

This paper combines the preprints [Au] and [CGR2]. Colbois and Ghanaat had the opportunity to work on [CGR2] at the Forschungsinstitut für Mathematik at ETH Zürich; they thank Marc Burger and the Forschungsinstitut for their hospitality and support. The four authors thank Christian Bär, Sylvestre Gallot and Chadwick Sprouse for helpful remarks.

2. Curvature and elliptic estimates

In this section we prepare general Harnack and regularity estimates for Schrödinger operators in a form suitable for the proof of Theorem 1.1. Such estimates have been obtained in [LCR] and [Ga1].

In what follows, (M, g) will be a compact connected Riemannian manifold of dimension n and diameter $\mathrm{diam}(M, g) \leq d$. We consider a Riemannian vector bundle E on M, equipped with a connection ∇ that is compatible with the fiber metric $\langle\ ,\ \rangle$. The inner product on E and the Riemannian measure μ on M are used to define normalized L^p–norms

$$\|S\|_p = \left(\frac{1}{\mathrm{vol}(M)} \int_M |S|^p \, d\mu\right)^{1/p}$$

for sections $S \in L^p(E)$ of E, as well as Sobolev norms on the corresponding spaces $L^p_k(E)$ of sections with p–integrable k–th covariant derivatives. Hölder's inequality implies that $\|S\|_q \leq \|S\|_p$ for $1 \leq q \leq p$.

Our estimates require certain Sobolev inequalities on M. The following version due to Gallot (see [Ga2] p. 203) is adequate for the present purpose. We note that suitable constants $C(n, p, q)$ and $\zeta(n, p, q)$ can be determined explicitly.

Lemma 2.1. *For every dimension n and every pair of real numbers $p \geq q > n$, there are constants $\zeta(n,p,q) > 0$ and $C(n,p,q)$ such that the following is true. If (M^n, g) is a compact Riemannian manifold such that* $\operatorname{diam}(M,g) \leq d$ *and*

$$\|\underline{\mathrm{Ric}}^-\|_{p/2}\, d^2 \leq \zeta(n,p,q),$$

then every function $u \in L^2_1(M)$ satisfies

$$\|u\|_{2q/(q-2)} \leq C(n,p,q)d\, \|du\|_2 + \|u\|_2, \tag{2.1}$$

and every $u \in L^q_1(M)$ satisfies

$$\sup u - \inf u \leq C(n,p,q)d\, \|du\|_q. \tag{2.2}$$

The *rough Laplacian* operating on sections of E is defined as $\bar{\Delta} S = \nabla^* \nabla S$. Here ∇^* is the adjoint of ∇ with respect to the L^2–inner product. We consider *Schrödinger operators* of the form $\bar{\Delta} + V$, where the potential $V \in C^\infty(\mathrm{Sym}(E))$ is a smooth field of symmetric endomorphisms of E. Weitzenböck's formula (1.1) shows that the de Rham–Laplacian Δ on 1-forms is an operator of this type.

The next result is a regularity estimate for linear combinations of eigensections of Schrödinger operators. For $m \in M$, let $\nu(m)$ be the lowest eigenvalue of $V(m)$ acting on the fiber E_m. Consider an L^2–orthonormal system S_i, $(i = 1, 2, \ldots)$ of eigensections of $\bar{\Delta} + V$. For a finite set I of positive integers, let $F_I \leq L^2(E)$ be the vector space spanned by $\{S_i \mid i \in I\}$. As before, f^- denotes the negative part of a real valued function f.

Theorem 2.2. *For every integer $n \geq 2$ and all real numbers $p > q > n$, there are explicit constants $\zeta(n,p,q) > 0$ and $a(n,p,q)$ such that the following is true. Suppose (M^n, g) satisfies* $\operatorname{diam}(M,g) \leq d$ *and*

$$\|\underline{\mathrm{Ric}}^-\|_{p/2}\, d^2 \leq \zeta(n,p,q).$$

Then for every $S \in F_I$ and $\kappa \in \mathbf{R}$ we have

$$\|S\|_\infty \leq \left(1 + a(n,p,q)\frac{\Lambda}{1+\Lambda}\right)(1+\Lambda)^{pq/(2(p-q))}\, \|S\|_2, \tag{2.3}$$

where

$$\Lambda = \left(\, \|(\nu - \kappa)^-)\|_{p/2} + \sup\nolimits_{i \in I} |\lambda_i - \kappa|\, \right)^{1/2} d.$$

Proof. For $k \in [1, \infty]$ define

$$A_k = A_k(I) = \sup \{\, \|S\|_k / \|S\|_2 \mid S \in F_I - \{0\} \,\}.$$

Then A_k is increasing as a function of k. We shall use the classical Moser iteration method to obtain an upper bound for A_∞.

Let $S \in F_I$ and $\kappa \in \mathbf{R}$. Fix $\varepsilon > 0$ and let $f = \sqrt{|S|^2 + \epsilon^2}$. The Cauchy–Schwarz inequality implies that

$$|df|^2 \leq \frac{|\nabla S|^2 |S|^2}{|S|^2 + \epsilon^2} \leq |\nabla S|^2,$$

and therefore

$$\begin{aligned} f\Delta f &= \frac{1}{2}\Delta(f^2) + |df|^2 \\ &\le \frac{1}{2}\Delta(|S|^2) + |\nabla S|^2 \\ &= \langle\, \bar\Delta S, S \,\rangle \\ &\le \big\langle\, (\bar\Delta + V - \kappa)S,\, S \,\big\rangle + (\nu-\kappa)^- |S|^2 \\ &\le |(\bar\Delta + V - \kappa)S|\, f + (\nu-\kappa)^- f^2. \end{aligned}$$

So for any real $k > 1/2$,

$$\begin{aligned} \int_M |d(f^k)|^2 &= \frac{k^2}{2k-1}\int_M \langle df,\, d(f^{2k-1})\rangle \\ &= \frac{k^2}{2k-1}\int_M (\Delta f) f^{2k-1} \\ &\le \frac{k^2}{2k-1}\left(\int_M |(\bar\Delta + V - \kappa)S|\, f^{2k-1} + \int_M (\nu-\kappa)^- f^{2k}\right) \end{aligned}$$

and using Hölder's inequality we obtain

$$\|d(f^k)\|_2^2 \le \frac{k^2}{2k-1}\left(\|(\nu-\kappa)^-\|_{p/2}\, \|f\|_{2kp/(p-2)}^{2k} + \|(\bar\Delta + V - \kappa)S\|_{2k}\, \|f\|_{2k}^{2k-1}\right).$$

The Sobolev inequality (2.1), applied to the function $u = f^k$ yields an estimate on the norm $\|u\|_{2q/(q-2)} = \|S\|_{2kq/(q-2)}^k$. Letting ε tend to zero we get

$$\begin{aligned} &\|S\|_{2kq/(q-2)}^k \\ &\le \|S\|_{2k}^k + \frac{Ckd}{\sqrt{2k-1}}\left(\|(\nu-\kappa)^-\|_{p/2}\, \|S\|_{2kp/(p-2)}^{2k} + \|(\bar\Delta + V - \kappa)S\|_{2k}\, \|S\|_{2k}^{2k-1}\right)^{1/2} \end{aligned}$$

with a constant $C = C(n,p,q)$ depending only on the quantities indicated. Since $\bar\Delta + V - \kappa$ maps F_I into itself, we have

$$\begin{aligned} \|(\bar\Delta + V - \kappa)S\|_{2k} &\le A_{2k}\, \|(\bar\Delta + V - \kappa)S\|_2 \\ &\le A_{2kp/(p-2)}\, \sup_{i\in I} |\lambda_i - \kappa|\, \|S\|_2, \end{aligned}$$

and $\|S\|_{2k} \le \|S\|_{2kp/(p-2)} \le A_{2kp/(p-2)}\, \|S\|_2$ then implies

$$\|S\|_{2kq/(q-2)} \le \left(1 + \frac{Ck\Lambda}{\sqrt{2k-1}}\right)^{1/k} A_{2kp/(p-2)}\, \|S\|_2.$$

This is true for every $S \in F_I$, so we get

$$A_{2kq/(q-2)} \le \left(1 + \frac{Ck\Lambda}{\sqrt{2k-1}}\right)^{1/k} A_{2kp/(p-2)}$$

for every $k > 1/2$. We use this inequality for $k = \beta^j$ with $\beta = \dfrac{q(p-2)}{p(q-2)} > 1$ and with $j = 0, 1, 2, \dots$ to obtain first

$$A_{2p\beta^m/(p-2)} \leq \prod_{j=0}^{m-1} \left(1 + \frac{C\beta^j \Lambda}{\sqrt{2\beta^j - 1}}\right)^{\beta^{-j}} A_{2p/(p-2)},$$

and then, by taking the limit as m tends to infinity,

$$A_\infty \leq \prod_{j=0}^{\infty} \left(1 + \frac{C\beta^j \Lambda}{\sqrt{2\beta^j - 1}}\right)^{\beta^{-j}} A_{2p/(p-2)}.$$

The inequality $\|S\|_{2p/(p-2)} \leq \|S\|_\infty^{2/p} \, \|S\|_2^{(p-2)/p}$ translates into

$$A_{2p/(p-2)} \leq A_\infty^{2/p}$$

and we obtain

$$A_\infty \leq \prod_{j=0}^{\infty} \left(1 + \frac{C\beta^j \Lambda}{\sqrt{2\beta^j - 1}}\right)^{\beta^{-j} p/(p-2)}. \tag{2.4}$$

To simplify this estimate, we note that $\ln(1 + ax) < \ln(1 + x) + ax/(x+1)$, provided a and x are positive. Therefore,

$$\begin{aligned} \ln A_\infty &\leq \frac{p}{p-2} \sum_{j=0}^{\infty} \frac{1}{\beta^j} \left(\ln(1+\Lambda) + C\beta^{j/2} \frac{\Lambda}{1+\Lambda}\right) \\ &= \frac{pq}{2(p-q)} \ln(1+\Lambda) + C'(n,p,q) \frac{\Lambda}{1+\Lambda}. \end{aligned}$$

For any $x \in [0, 1]$, we have $e^{ax} \leq ax\, e^a + 1$, and we conclude that

$$A_\infty \leq \left(1 + a(n,p,q) \frac{\Lambda}{1+\Lambda}\right) (1+\Lambda)^{pq/(2(p-q))}.$$

Lemma 2.3. *Every smooth section S of E satisfies the pointwise inequality*

$$\begin{aligned} \frac{1}{2}\Delta(|\nabla S|^2) + |\nabla^2 S|^2 \leq \langle \nabla^* R^E S, \nabla S\rangle + \underline{\mathrm{Ric}}^- |\nabla S|^2 \\ + \langle \nabla \bar{\Delta} S, \nabla S\rangle + |R^E| \, |\nabla S|^2, \end{aligned} \tag{2.5}$$

where, for vector fields X and Y on M, $R^E_{X,Y} = \nabla^2_{X,Y} - \nabla^2_{Y,X}$ is the curvature of E and, in abstract index notation,

$$\langle \nabla^* R^E S, \nabla S\rangle := \left\langle -\nabla_j (R^E_{ij} S), \nabla_i S\right\rangle. \tag{2.6}$$

Proof. A standard calculation interchanging covariant derivatives shows that

$$\frac{1}{2}\Delta(|\nabla S|^2) + |\nabla^2 S|^2 = \langle \nabla^* R^E S, \nabla S\rangle - \mathrm{Ric}_{ij}\ \langle \nabla_i S, \nabla_j S\rangle + \langle \nabla \bar{\Delta} S, \nabla S\rangle + \left\langle R^E_{ij}(\nabla_j S), \nabla_i S\right\rangle .$$

Lemma 2.3 is an immediate consequence.

The following Harnack inequality is essentially due to Le Couturier and Robert. It is stated as Theorem 1.4.1 in [LCR] for solutions S of a Schrödinger equation $(\bar{\Delta} + V)S = 0$. We note that the hypothesis $q \geq 4$ was omitted in [LCR].

Theorem 2.4. *For every dimension $n \geq 2$ and real number $p > n$, there are explicit constants $\zeta(n,p) > 0$ and $A(n,p)$ such that the following holds. If (M,g) satisfies*

$$\|\underline{\mathrm{Ric}}^-\|_{p/2}\, d^2 \leq \zeta(n,p),$$

then for every smooth section S of E

$$1 - \frac{\inf |S|}{\sup |S|} \leq A(n,p) \left(\frac{\|\nabla S\|_2\, d}{\|S\|_\infty}\right)^{\tau} \left(1 + \|R^E\|_{q/2}\, d^2 + \frac{\|\bar{\Delta} S\|_{q/2}\, d^2}{\|S\|_\infty}\right)^{1-\tau},$$

where $q = \max\{p, 4\}$ and $\tau = \dfrac{2(p-n)}{pq + n(q-4)}$. In particular, $0 < \tau < 1$.

Proof. The idea of the proof (see [LCR]) is to apply Lemma 2.3 and the Sobolev inequality (2.1) to obtain a bound on a suitable integral norm of ∇S. Then inequality (2.2) yields the result.

Let $u = \sqrt{|\nabla S|^2 + \epsilon^2}$. As in the proof of Theorem 2.2, we obtain that

$$u\Delta u \leq \frac{1}{2}\Delta|\nabla S|^2 + |\nabla^2 S|^2.$$

This inequality and Lemma 2.3 imply

$$\begin{aligned}\int_M |d(u^k)|^2 &= \frac{k^2}{2k-1}\int_M \langle du, d(u^{2k-1})\rangle \\ &= \frac{k^2}{2k-1}\int_M u\Delta u\, u^{2k-2} \\ &\leq \frac{k^2}{2k-1}\int_M \Big(\underline{\mathrm{Ric}}^- u^{2k} + \langle \nabla\bar{\Delta} S, \nabla S\rangle\, u^{2k-2} \\ &\qquad + \langle \nabla^* R^E S, \nabla S\rangle\, u^{2k-2} + |R^E|\, u^{2k}\Big).\end{aligned}$$

The relation $|ab| \leq a^2 + \dfrac{b^2}{4}$ and the divergence theorem applied to the vector field

$$u^{2k-2}\langle \bar{\Delta} S, \nabla_{.} S\rangle^\sharp$$

imply that for $k \geq 1$

$$\begin{aligned}\int_M \langle \nabla \bar{\Delta} S, \nabla S \rangle \, u^{2k-2} &= \int_M |\bar{\Delta} S|^2 \, u^{2k-2} - (2k-2) \int_M \langle \bar{\Delta} S, \nabla_{\mathrm{grad}\, u} S \rangle \, u^{2k-3} \\ &\leq \int_M |\bar{\Delta} S|^2 \, u^{2k-2} + (2k-2) \int_M |\bar{\Delta} S| \, |du| \, u^{2k-2} \\ &\leq \frac{k-1}{2} \int_M |du|^2 \, u^{2k-2} + (2k-1) \int_M |\bar{\Delta} S|^2 \, u^{2k-2}.\end{aligned}$$

From the skew symmetry $R^E_{X,Y} = -R^E_{Y,X}$ we have

$$\textstyle\sum_{i,j} \left\langle R^E_{ij} S, \nabla^2_{ij} S \right\rangle = \frac{1}{2} \, |R^E S|^2.$$

Because of (2.6), the divergence theorem applied to the vector field

$$\textstyle\sum_i u^{2k-2} \langle R^E_{(i,\cdot)} S, \nabla_i S \rangle^\sharp$$

yields

$$\begin{aligned}\int_M \langle \nabla^* R^E S, \nabla S \rangle \, u^{2k-2} &\leq \frac{k-1}{2} \int_M |du|^2 u^{2k-2} + (2k-1) \int_M |R^E S|^2 \, u^{2k-2} \\ &= \frac{k-1}{2k^2} \int_M \left| d(u^k) \right|^2 + (2k-1) \int_M |R^E S|^2 \, u^{2k-2}.\end{aligned}$$

If we set $k = (q-2)/2$ and apply Hölder's inequality, we obtain for $q \geq 4$

$$\begin{aligned}\|d(u^{(q-2)/2})\|_2 \leq \frac{q-2}{\sqrt{2}} \Bigg(&\left(\|(\underline{\mathrm{Ric}}^- \|_{p/2} + \|R^E\|_{p/2} \right) \|u\|^{q-2}_{p(q-2)/(p-2)} \\ &+ \left(\|\bar{\Delta} S\|^2_{q/2} + \|R^E S\|^2_{q/2} \right) \|u\|^{q-4}_q \Bigg)^{1/2}\end{aligned}$$

Now let $p > n$ and $r = (p+n)/2$. Hölder's inequality implies that

$$\|u\|^{\rho+\sigma}_b \leq \|u\|^\rho_a \, \|u\|^\sigma_c$$

if $1 \leq a \leq b \leq c$, and if ρ and σ are positive numbers such that $(\rho+\sigma)/b = \rho/a + \sigma/c$. We apply this inequality, and use the Sobolev inequality (2.1) for the function $u^{(q-2)/2}$. Setting $\gamma := \frac{2(p-r)(q-2)}{(q-4)pr+4r}$ we obtain that for $q \geq 4$

$$\begin{aligned}\|u\|^{-\gamma}_2 \, \|u\|^{\gamma+(q-2)/2}_{p(q-2)/(p-2)} &\leq \|u\|^{(q-2)/2}_{(q-2)r/(r-2)} = \|u^{(q-2)/2}\|_{2r/(r-2)} \\ &\leq \|u\|^{(q-2)/2}_{q-2} + C'(n,p) \, d \Bigg(\left(\|(\underline{\mathrm{Ric}}^- \|_{p/2} + n \, \|R^E\|_{p/2} \right) \|u\|^{q-2}_{p(q-2)/(p-2)} \\ &\qquad + \left(\|\bar{\Delta} S\|^2_{q/2} + \|R^E S\|^2_{q/2} \right) \|u\|^{q-4}_q \Bigg)^{1/2}\end{aligned}$$

If we set $q = \max\{p, 4\}$, then $\gamma = 2(p-r)/(r(q-2))$ and $p(q-2)/(p-2) \geq q \geq p$. Letting ε tend to zero, we get

$$\|\nabla S\|^\gamma_2 \geq f \left(\|\nabla S\|_{p(q-2)/(p-2)} \right) \tag{2.7}$$

where f is given by

$$f(x)=\frac{x^{\gamma+1}}{x+C'(n,p)d\sqrt{ax^2+b}}$$

with

$$a=\|\underline{\mathrm{Ric}}^-\|_{p/2}+\|R^E\|_{p/2}\quad\text{and}$$
$$b=\|\bar\Delta S\|_{q/2}^2+\|R^E S\|_{q/2}^2.$$

Since the function f is increasing, we can replace $\|\nabla S\|_{p(q-2)/(p-2)}$ by $\|\nabla S\|_p$ on the right hand side of (2.7), and then, using the Sobolev inequality (2.2), by $(\sup|S|-\inf|S|)\,/C(n,p)d$ to obtain

$$(\|\nabla S\|_2\, d)^\gamma\geq\frac{(\sup|S|-\inf|S|)^{1+\gamma}}{C''(n,p)\left(\|S\|_\infty+\sqrt{a\|S\|_\infty^2 d^2+bd^4}\right)}$$

with a new constant $C''(n,p)$. By hypothesis, we have $\|\underline{\mathrm{Ric}}^-\|_{p/2}\, d^2\leq 1$. Theorem 2.4 follows using

$$\|R^E S\|_{q/2}^2\leq\|R^E\|_{q/2}^2\,\|S\|_\infty\quad\text{and}$$
$$\|R^E\|_{p/2}d^2\leq 1+\|R^E\|_{q/2}^2 d^4.$$

3. Proof of Theorem 1.1

The estimates obtained in section 2 can be applied to the cotangent bundle $E=T^*M$ of a Riemannian manifold, together with its de Rham–Laplacian $\Delta=\bar\Delta+V$ on one–forms α. In this case, the curvature R^E is the curvature tensor R of M acting on forms, and the potential is given by $V(\alpha)=\mathrm{Ric}(\alpha^\sharp,\cdot)$. In particular, the lower bound ν is equal to $\underline{\mathrm{Ric}}^-$. We choose the index set $I=\{1,\dots,k\}$ and let $q=(p+n)/2$ in Theorem 2.2. Then by inequality (2.3), linear combinations α of eigenforms corresponding to the first k eigenvalues $\lambda_1,\dots,\lambda_k$ satisfy

$$\|\alpha\|_\infty\leq c_1\,\|\alpha\|_2,\tag{3.1}$$

where

$$c_1=\left(1+a(n,p)\frac{\Lambda}{1+\Lambda}\right)(1+\Lambda)^{p(p+n)/(2(p-n))}$$
$$\leq a_1(n,p)(1+\Lambda)^{p(p+n)/(2(p-n))}$$
$$\Lambda=\left(\,\|\underline{\mathrm{Ric}}^-\|_{p/2}+\lambda_k\,\right)^{1/2}d.$$

On the other hand, Theorem 2.4 yields an estimate

$$\begin{aligned}&\sup|\alpha|-\inf|\alpha|\\&\quad\leq A(n,p)\,(\|\nabla\alpha\|_2\, d)^\tau\left(\,(1+\|R\|_{q/2}\, d^2)\,\|\alpha\|_\infty+\|\bar\Delta\alpha\|_{q/2}\, d^2\,\right)^{1-\tau}\end{aligned}\tag{3.2}$$

for every smooth one–form α, where $q=\max\{p,4\}$, and where $0<\tau<1$ depends only on n and p.

It is sufficient to prove Theorem 2.1 under the assumption that $d = 1$, as the general case is then obtained by rescaling the metric. We will also assume that

$$\Lambda^2 = \|\underline{\mathrm{Ric}}^-\|_{p/2} + \lambda_k \le \varepsilon_0 \left(1 + \|R\|_{q/2}\right)^{-\beta(n,p)} \tag{3.3}$$

and then impose restrictions of the form $\varepsilon_0 \le \varepsilon(n,p)$ consistent with our hypotheses (1.5) and (1.6). Let $\omega^1, \ldots, \omega^k$ be eigenforms corresponding to the first k eigenvalues $\lambda_1, \ldots, \lambda_k$, orthonormal with respect to the volume normalized L^2 inner product. Our goal ist to show that, for suitably small ε_0, the forms ω^i are nearly orthonormal at every point of M, and that their exterior derivatives are small in the maximum norm. First we apply (3.2) to $\alpha = \omega^i$. Weitzenböck's formula (1.1) shows that

$$\begin{aligned}
\|\nabla\omega^i\|_2^2 &= \lambda_i \|\omega^i\|_2^2 + \frac{1}{\mathrm{vol}(M)} \int_M (-\mathrm{Ric})(\omega^i, \omega^i) \\
&\le \frac{1}{\mathrm{vol}(M)} \int_M (\lambda_k - \underline{\mathrm{Ric}})\, |\omega^i|^2 \\
&\le \frac{1}{\mathrm{vol}(M)} \int_M (\underline{\mathrm{Ric}} - \lambda_k)^-\, |\omega^i|^2 \\
&\le \|\, (\underline{\mathrm{Ric}} - \lambda_k)^-\, \|_1\, \|\omega^i\|_\infty^2,
\end{aligned}$$

and (3.1) then implies

$$\|\nabla\omega^i\|_2 \le c_1\, \|\, (\underline{\mathrm{Ric}} - \lambda_k)^-\, \|_1^{1/2}\, . \tag{3.4}$$

Again using (1.1) and (3.1), we get

$$\begin{aligned}
\|\bar\Delta\omega^i\|_{q/2} &\le \|\Delta\omega^i\|_{q/2} + \|\mathrm{Ric}\|_{q/2}\, \|\omega^i\|_\infty \\
&\le (\lambda_k + \|\mathrm{Ric}\|_{q/2})\, \|\omega^i\|_\infty \\
&\le (\lambda_k + \|\mathrm{Ric}\|_{q/2}) c_1.
\end{aligned} \tag{3.5}$$

We substitute these inequalities into (3.2), assuming an initial restriction $\varepsilon_0 \le 1$, so that $\lambda_k \le 1$, to obtain

$$\begin{aligned}
&\sup|\omega^i| - \inf|\omega^i| \\
&\quad \le A(n,p) \left(c_1\, \|\, (\underline{\mathrm{Ric}} - \lambda_k)^-\, \|_1^{1/2}\right)^\tau \left((1 + \|R\|_{q/2}) c_1 + (\lambda_k + \|\mathrm{Ric}\|_{q/2}) c_1 \right)^{1-\tau} \\
&\quad \le 2^{1-\tau} c_1 A(n,p)\, \|(\underline{\mathrm{Ric}} - \lambda_k)^-\|_1^{\tau/2}\, (1 + \|R\|_{q/2})^{1-\tau}.
\end{aligned} \tag{3.6}$$

If we simplify this using $\|(\underline{\mathrm{Ric}} - \lambda_k)^-\|_1 \le \|\underline{\mathrm{Ric}}^-\|_1 + \lambda_k \le \Lambda^2$, the definition of c_1 from (3.1), and $\Lambda \le 1$, we get

$$\begin{aligned}
\sup|\omega^i| - \inf|\omega^i| &\le a_2(n,p)\, (1+\Lambda)^{p(p+n)/(2(p-n))} \Lambda^\tau (1 + \|R\|_{q/2})^{1-\tau} \\
&\le a_3(n,p)\, \Lambda^\tau (1 + \|R\|_{q/2})^{1-\tau} \\
&\le a_3(n,p)\, \varepsilon_0^{\tau/2}.
\end{aligned}$$

As a consequence,

$$\sup|\omega^i| - \inf|\omega^i| \le \varepsilon_1, \tag{3.7}$$

where $\varepsilon_1 = a_3(n,p)\varepsilon_0^{\tau/2}$ can be made as small as we wish by requiring that $\varepsilon_0 \leq \varepsilon(n,p)$ for suitably small $\varepsilon(n,p)$.

Since $\|\omega^i\|_2 = 1$, there are points in M where $|\omega^i| = 1$, and we obtain

$$1 - \varepsilon_1 \leq |\omega^i(p)| \leq 1 + \varepsilon_1$$

for every $p \in M$. We now consider the inner products $\langle \omega^i, \omega^j \rangle$ for $i \neq j$. Applying (3.2) as before, but now to $\alpha = \omega^i - \omega^j$ instead of ω^i, we obtain, using the triangle inequality

$$\sup |\omega^i - \omega^j| - \inf |\omega^i - \omega^j| \leq 4\,\varepsilon_1.$$

Since $\|\omega^i - \omega^j\|_2 = \sqrt{2}$, there are points in M where $|\omega^i - \omega^j| = \sqrt{2}$, and so

$$\sqrt{2} - 4\,\varepsilon_1 \leq |\omega^i(p) - \omega^j(p)| \leq \sqrt{2} + 4\,\varepsilon_1$$

for $p \in M$. Using

$$2\langle \omega^i, \omega^j \rangle = |\omega^i|^2 + |\omega^j|^2 - |\omega^i - \omega^j|^2,$$

we obtain

$$-10\,\varepsilon_1 \leq \langle \omega^i, \omega^j \rangle \leq 10\,\varepsilon_1.$$

We have shown that there is a pointwise inequality

$$|\langle \omega^i, \omega^j \rangle - \delta^{ij}| \leq \varepsilon_2, \tag{3.8}$$

for $i, j = 1, \ldots, k$, where $\varepsilon_2 = 10\,\varepsilon_1 = 10\,a_3(n,p)\,\varepsilon_0^{\tau/2}$.

For $\varepsilon_2 < 1/n$, it follows that the ω^i are linearly independent everywhere, so that TM has a trivial subbundle of rank k. In particular, we have $k \leq n$. And if $k = n$, then $\lambda_{n+1} > \lambda_n$.

We now consider the case $k = n$. Then $\omega = (\omega^1, \ldots, \omega^n)$ is a coframe. Also, by (3.8), the Riemannian metric

$$g_\omega = \sum\nolimits_{i=1}^n \omega^i \otimes \omega^i$$

induced by ω on M is close to the original metric g. We apply the following result from [Gh1] (see Theorem A in [Gh2] for a more detailed statement).

Theorem 3.1. *There is a constant $\varepsilon(n) > 0$ such that the following is true. If M is a compact n–manifold with a coframe $\omega : TM \to \mathbf{R}^n$ whose exterior derivative satisfies $\|d\omega\|_\infty d < \varepsilon(n)$, where $\operatorname{diam}(M, g_\omega) \leq d$, then M is diffeomorphic to a nilmanifold $\Gamma\backslash G$. One can choose a Maurer–Cartan form ω_0 on $\Gamma\backslash G$ and a diffeomorphism $\phi : M \to \Gamma\backslash G$ such that*

$$\|\omega - \phi^*\omega_0\|_\infty \leq c(n)\,\|d\omega\|_\infty d, \tag{3.9}$$

with a constant $c(n)$ depending only on n.

In this result, the norms and diameter are with respect to g_ω. Because of (3.8), the difference between g_ω and the given g is negligible for our purpose.

In order to use Theorem 3.1, we need a bound on $\|d\omega^i\|_\infty$ analogous to (3.1) for the two–forms $\alpha = d\omega^i$. We apply Theorem 2.2 to the bundle of two–forms and its de Rham–Laplacian. The corresponding Weitzenböck formula (see [Sa], p. 303) has the form $\Delta = \bar{\Delta} + V$, where the potential V satisfies $\|\nu^-\|_{p/2} \leq c(n)\|R\|_{p/2}$. As in (3.1), Theorem 2.2 with $\kappa = 0$ then yields

$$\|d\omega^i\|_\infty \leq c_2\|d\omega^i\|_2 \leq c_2\sqrt{\lambda_n}, \tag{3.10}$$

where

$$c_2 \leq a_3(n,p)(1+\Lambda)^{p(p+n)/(2(p-n))},$$

but this time

$$\Lambda = \left(\|R\|_{p/2} + \lambda_n \right)^{1/2}.$$

Theorem 1.1 under hypothesis (1.6) is an immediate consequence.

Now consider the case of $n-1$ small eigenvalues, hypothesis (1.7). We first show that if M is orientable, then the n–th eigenvalue is also small, so that we can apply the result we already proved. Choose L^2–orthonormal eigenforms $\omega^1, \ldots, \omega^{n-1}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Since M is orientable, we can use the Hodge star operator to define

$$\eta = *(\omega^1 \wedge \ldots \wedge \omega^{n-1}). \tag{3.11}$$

This form is L^2–orthogonal (in fact pointwise orthogonal) to $\omega^1, \ldots, \omega^{n-1}$. By the minimax principle, its Rayleigh quotient,

$$R(\eta) = \frac{\|d\eta\|_2^2 + \|\delta\eta\|_2^2}{\|\eta\|_2^2}$$

is an upper bound for λ_n. In terms of a local orthonormal frame field $e_1, \ldots, e_n$ for M, the codifferential of η is given by $\delta\eta = -\iota_{e_i}\nabla_{e_i}\eta$. Therefore, using (3.4),

$$\begin{aligned}\|d\eta\|_2 &= \|\delta(\omega^1 \wedge \ldots \wedge \omega^{n-1})\|_2 \\ &\leq c(n) \sum\nolimits_i \left(\|\nabla\omega^i\|_2 \prod\nolimits_{j\neq i} \|\omega^i\|_\infty \right) \\ &\leq c(n)\, c_1^{n-1} \left(\|\underline{\mathrm{Ric}}^-\|_1 + \lambda_k \right)^{1/2}.\end{aligned}$$

On the other hand, $\|d\omega^i\|_2 \leq \|\nabla\omega^i\|_2$ and again (3.4) imply

$$\begin{aligned}\|\delta\eta\|_2 &= \|d(\omega^1 \wedge \ldots \wedge \omega^{n-1})\|_2 \\ &\leq \sum\nolimits_i \|d\omega^i\|_2 \|\omega^1 \wedge \ldots \wedge \widehat{\omega^i} \wedge \ldots \wedge \omega^{n-1}\|_\infty \\ &\leq (n-1)\, c_1^{n-1} \left(\|\underline{\mathrm{Ric}}^-\|_1 + \lambda_k \right)^{1/2}.\end{aligned}$$

Inequality (3.8) yields a lower bound on the denominator of the Rayleigh quotient, and we obtain

$$\lambda_n \leq R(\eta) \leq (1 + c(n)\,\varepsilon_2)\; c(n)\, c_1^{n-1} \left(\|\underline{\mathrm{Ric}}^-\|_1 + \lambda_k \right)^{1/2}, \tag{3.12}$$

which is the required smallness of λ_n.

If M is not orientable, then our argument implies that the twofold orientable covering $\tilde{M}$ is a nilmanifold. This is not quite sufficient, as we need to show that the deck transformation $\sigma : \tilde{M} \to \tilde{M}$ is an isometry of some left invariant Riemannian metric on $\tilde{M}$. We therefore proceed as follows.

The pullbacks $\tilde{\omega}^1, \ldots, \tilde{\omega}^{n-1}$ of the eigenforms to $\tilde{M}$ are eigenforms of the pullback metric $\tilde{g}$. By (3.8), these forms are almost orthonormal at every point. Considering the Rayleigh quotient for the form $*(\tilde{\omega}^1 \wedge \ldots \wedge \tilde{\omega}^{n-1})$ as before, we conclude that the n–th eigenvalue $\tilde{\lambda}_n$ of $\tilde{M}$ is small. Now let $\tilde{\omega}^n$ be an eigenform corresponding to $\tilde{\lambda}_n$, such that $\tilde{\omega}^1, \ldots, \tilde{\omega}^n$ are L^2–orthonormal. Our previous argument shows that the $\mathbf{R}^n$–valued one–form $\tilde{\omega} = (\tilde{\omega}^1, \ldots, \tilde{\omega}^n)$ is a coframe with small exterior derivative. The transformation σ leaves the sum of the eigenspaces for the first n eigenvalues invariant, because it is an isometry of $\tilde{g}$ and since $\tilde{\lambda}_{n+1} > \tilde{\lambda}_n$. Therefore, σ acts as an *affine isometry* of the coframe $\tilde{\omega}$, i.e. satisfies

$$\sigma^* \tilde{\omega} = a \circ \tilde{\omega}$$

for some constant orthogonal map a of $\mathbf{R}^n$. By [Gh2], affine isometries of $\tilde{\omega}$ are affine isometries of the Maurer–Cartan coframe $\phi^* \omega_0$ in Theorem 3.1 as well, and the proof is complete.

4. Anderson's examples

In this section, we show that Theorem 1.1 does not hold without an assumption involving the full curvature tensor R. More precisely, we have

Proposition 4.1. *For every dimension $n \geq 4$, there are closed n–manifolds with arbitrary large second real Betti numbers b_2 that admit Riemannian metrics of diameter one with $\|\mathrm{Ric}\|_\infty + \lambda_n \leq \varepsilon$ for any given $\varepsilon > 0$.*

Nomizu's theorem mentioned in Remark 1.2(iv) implies that manifolds with $b_2 > n(n-1)/2$ are not homotopy equivalent to infra–nilmanifolds. The examples we use have been constructed by Anderson. Theorem 0.4 of [An] exhibits manifolds M with diameter one, $\|\mathrm{Ric}\|_\infty \leq \varepsilon$, first Betti number $b_1 = n-1$ (so that $\lambda_{n-1} = 0$), and b_2 arbitrary large. This shows that our result on λ_{n-1} does not hold if we replace (1.5),(1.6) by conditions not involving R.

For the *proof* of Proposition 5.1, we now describe Anderson's simplest examples in more detail (see p. 73 and Remark 2.1 on p. 75 of [An]). These examples are obtained by performing surgery killing a generator of the fundamental group of a flat n–torus. One starts with a Riemannian product $M_0 = S^1_\delta \times T^{n-1}$ of a circle of suitable diameter δ and a flat $(n-1)$–torus. Removing a subset of the form $S^1_\delta \times B_a$, where $B_a \subseteq T^{n-1}$ is a ball of radius a less than half the injectivity radius of T^{n-1}, we are left with $M_1 = S^1_\delta \times \left(T^{n-1} \backslash B_a\right)$, a manifold with boundary $S^1_\delta \times S^{n-2}_a$. To this one glues a copy of $D^2 \times S^{n-2}$, carrying a suitably scaled Riemannian Schwarzschild metric g_{aR}, to obtain

$$M = \left(S^1_\delta \times (T^{n-1} \backslash B_a)\right) \cup \left(D^2 \times S^{n-2}\right)$$

equipped with a Riemannian metric $g_{a,R}$ that has diameter less than $n\pi$ and satisfies $\|\mathrm{Ric}\|_\infty \to 0$ as $aR \to \infty$. The construction involves parameters a, R and δ that need to be related by $\delta = c_0/aR$, where c_0 depends only on the dimension n. We may choose the radius $a = R^{-1/2}$ and then let R tend to infinity to obtain metrics g_R satisfying $\|\mathrm{Ric}\|_\infty \le \varepsilon$ and $\mathrm{diam}(M, g_R) \le n\pi$, that coincide with our original flat metric on M_0 outside of $S^1_\delta \times B_a$. Here ε as well as a can be made as small as required by choosing R large.

We now claim that the n–th eigenvalue $\lambda_n(g_R)$ tends to zero as $R \to \infty$. To show this, we exhibit n one–forms $\beta^1, \dots, \beta^n$ on M that are almost orthonormal in $L^2(M)$ and whose Rayleigh quotients tend to zero. The minimax principle then implies our claim. The forms β^i are obtained as follows (see for example [RT]). Let $\varphi \in C^\infty(T^{n-1})$ be a function such that $0 \le \varphi \le 1$, $\varphi = 0$ on B_a, $\varphi = 1$ outside B_{2a}, and such that its differential satisfies $\|d\varphi\| \le 1/3a$. Since the dimension $n-1 > 2$, the L^2-norm $\|d\varphi\|_2 = O(a)$ as a tends to zero. Define a cutoff function ψ on $M_0 = S^1_\delta \times T^{n-1}$ by $\psi(t,x) = \varphi(x)$. We choose an L^2–orthonormal basis $\alpha^1, \dots, \alpha^n$ for the harmonic forms on our flat torus M_0, and let $\tilde\alpha^i = \varphi\alpha^i$. The forms α^i are in fact parallel and, if we use volume normalized L^2 inner products as in section 4, pointwise orthonormal. For the exterior derivatives we obtain

$$\|d\tilde\alpha^i\|_2 = \|d\varphi \wedge \alpha^i\|_2 \le \|d\varphi\|_2 \, \|\alpha^i\|_\infty,$$

and this tends to zero if a does. A corresponding inequality holds for $\|\delta\tilde\alpha^i\|_2$, while $\|\tilde\alpha^i\|_2 = \|\varphi\|_2$ tends to one. Therefore, the Rayleigh quotients of the $\tilde\alpha^i$ converge to zero with a. Since these forms vanish on the domain affected by our surgery, they can be transplanted to M extending by zero on $D^2 \times S^{n-2}$ without changing their L^2 norms or Rayleigh quotients. In this way, we obtain the required forms β^i on M. Finally, this operation can be repeated on several disjoint balls B_a to yield examples with arbitrary large second Betti numbers.

5. Spectra of infra–nilmanifolds

Theorem 1.1 can be illustrated by the well known spectral behavior of *almost flat* left invariant metrics on infra–nilmanifolds (see [Gh1] p. 68). The left invariant Maurer–Cartan form $\omega : TG \to \mathcal{G}$ of a Lie group G descends to a coframe $\omega : T\bar M \to \mathcal{G}$ on any left quotient $\bar M = \Gamma\backslash G$ of G by a discrete subgroup. Inner products on the Lie algebra $\mathcal{G}$ imply, via ω, left invariant Riemannian metrics on $\bar M$. If, in particular, $\bar M$ is a compact nilmanifold, then there are families of inner products on $\mathcal{G}$ such that the corresponding left invariant metrics g_ε, $\varepsilon > 0$ have the properties that $\|d\omega\|_{\infty, g_\varepsilon} \to 0$ and $\mathrm{diam}(\bar M, g_\varepsilon) \to 0$ as ε tends to zero (see [BuK] p. 126), and standard formulas then imply that R also tends to zero.

Take an orthonormal basis for $\mathcal{G}$ with respect to g_ε and decompose $\omega = (\omega^1_\varepsilon, \dots, \omega^n_\varepsilon)$ accordingly. Since G is unimodular, we have $\delta\omega^i_\varepsilon = 0$, and the Rayleigh quotients for ω^i_ε are

$$\frac{\|(d+\delta)\omega^i_\varepsilon\|^2_{2,g_\varepsilon}}{\|\omega^i_\varepsilon\|^2_{2,g_\varepsilon}} = \|d\omega^i_\varepsilon\|^2_{\infty,g_\varepsilon} \to 0 \qquad (\varepsilon \to 0).$$

As a consequence, $\lambda_n(g_\varepsilon)$ tends to zero with ε, while inequalities (1.3) and (1.4) show that $\lambda_{n+1} d^2$ admits a lower bound converging to $\pi^2/32(n+1)^2$. We summarize these remarks in

Proposition 5.1. *Every compact n–dimensional nilmanifold admits families g_ε ($\varepsilon > 0$) of Riemannian metrics such that the diameter* $\mathrm{diam}(g_\varepsilon)$*, the curvature tensor* $\|R(g_\varepsilon)\|_{\infty,g_\varepsilon}$ *and the eigenvalue* $\lambda_n(g_\varepsilon)$ *converge to zero as* $\varepsilon \to 0$.

By Theorem 1.1, such metrics do not exist on infra–nilmanifolds that are not nilmanifolds. All compact infra–nilmanifolds are obtained as quotients $M = F\backslash\bar{M}$ of nilmanifolds $\bar{M}$ by finite groups F of *affine transformations*, and one can see how the small eigenvalues of $\bar{M}$ are lost when passing to the quotient. In fact, affine transformations φ are characterized by the property that $\varphi^*\omega = a \circ \omega$ for some constant linear map $a = \mathrm{rot}(\varphi) : \mathcal{G} \to \mathcal{G}$, the *rotational part* of φ. Left translations by elements of G are those affine transformations that have $\mathrm{rot}(\varphi) = 1$. So unless M is itself a nilmanifold, dividing by F will eliminate some of the small eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\bar{M}$. And it will eliminate all of them, when $\mathrm{rot}(F)$ acts irreducibly on $\mathcal{G}$.

This is the case for the three dimensional euclidean space forms labelled $\mathcal{G}_6$ on p. 122 of [Wo]. The first real Betti number of this space is zero, which means that all three of the small eigenvalues (corresponding to the harmonic forms) are lost when passing to $M = \mathcal{G}_6$ from its torus covering space $\bar{M}$.

The euclidean space form $\mathcal{G}_6$ provides an example for another problem in spectral theory. A theorem of Cheng ([Ch]) states that for closed Riemannian n–manifolds (M, g) with $\mathrm{Ric} \geq -(n-1)$ and diameter $\mathrm{diam}(M, g) \leq d$, the first non–zero eigenvalue $\lambda_2^{(0)}$ of the Laplacian on functions satisfies

$$\lambda_2^{(0)}(M, g) \leq \frac{(n-1)^2}{4} + \frac{c(n)}{d^2},$$

where $c(n)$ depends only on n. It may be asked (see [Lo]) whether results of this kind hold for the eigenvalues of the Laplacian on p–forms. For one–forms, this is true and a direct consequence of Cheng's result, by applying exterior differentiation to an eigenfunction corresponding to $\lambda_2^{(0)}$. However, the following counterexample shows that some caution is required for two–forms.

Example 5.2. Let $M = \mathcal{G}_6$ be as before. If we equip M with any flat metric g and consider the family $M_\varepsilon = (M, \varepsilon g)$, then as $\varepsilon \to 0$, all M_ε have zero curvature and diameter converging to zero. The first eigenvalue $\lambda_1^{(1)}(M_\varepsilon)$ of the Laplacian on one–forms, and by duality that on two–forms, tend to infinity.

Consider $N_\varepsilon = S^1 \times M_\varepsilon$ with S^1 a circle of diameter one. Then N_ε, endowed with the product metric, is a flat manifold with diameter converging to one as $\varepsilon \to 0$. The Künneth formula $\Delta(\alpha \wedge \beta) = \Delta\alpha \wedge \beta + \alpha \wedge \Delta\beta$ implies that the first eigenvalue of N_ε on two-forms is

$$\lambda_1^{(2)}(N_\varepsilon) = \min\left\{\lambda_1^{(0)}(S^1) + \lambda_1^{(2)}(M_\varepsilon),\ \lambda_1^{(1)}(S^1) + \lambda_1^{(1)}(M_\varepsilon)\right\},$$

which tends to infinity as $\varepsilon \to 0$. Rescaling the metric, we obtain a *family of compact flat four–manifolds of any given diameter d whose first eigenvalues on two-forms tend to infinity.*

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June 25, 2001