NON-PROPERNESS OF AMENABLE ACTIONS ON GRAPHS WITH INFINITELY MANY ENDS

Soyoung Moon and Alain Valette

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Abstract

We study amenable actions on graphs having infinitely many ends, giving a generalized answer to Ceccherini's question on groups with infinitely many ends.

1 Statement of the result

An action of a group G on a set X is *amenable* if there exists a G-invariant mean on X, i.e. a map $\mu : 2^X = \mathcal{P}(X) \to [0,1]$ such that $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$, for every disjoint subsets A, $B \subseteq X$, and $\mu(gA) = \mu(A)$, $\forall g \in G$, $\forall A \subseteq X$.

An isometric action of a group G on a metric space (X, d) is *proper* if for some $x_0 \in X$, and every $R > 0$, the set $\{g \in G \mid d(x_0, gx_0) \leq R\}$ is finite.

The aim of this note is to give a short proof of the following result:

Theorem 1. *Let $X = (V, E)$ be a locally finite graph with infinitely many ends. Let $\overline{X} = V \cup \partial X$ be the end compactification. Let G be a group of automorphisms of X. Assume that the action of G on V is amenable and there exists $x_0 \in V$ such that the orbit Gx_0 is dense in $\overline{X}$. Then there is a unique G-fixed end in ∂X, and the action of G (as a discrete group) on V is not proper.*

A deep result of Stallings [4] says that G has infinitely many ends if and only if G is an amalgamated free product $\Gamma_1 *_A \Gamma_2$ or HNN-extension $HNN(\Gamma, A, \varphi)$ with A finite (with $\min\{[\Gamma_1 : A], [\Gamma_2 : A]\} \geq 2$, not both 2, in the amalgamated product case; and $\min\{[\Gamma : A], [\Gamma : \varphi(A)]\} \geq 2$, not both 2, in the HNN case). In particular, if G has infinitely many ends, it contains non-abelian free subgroups, hence is non amenable. Tullio Ceccherini-Silberstein asked whether non-amenability of G could be proved without appealing to Stallings' theorem. Since a finitely generated group G with infinitely many ends acts properly and transitively on its Cayley graph, our result shows that G is not amenable.

Remarks

1. The density assumption of Theorem 1 is satisfied when G has finitely many orbits in V. This assumption is necessary; for example the action of $\mathbb{Z}$ on $\mathbb{F}_2 = \langle a, b \rangle$ defined by $n \cdot g = a^n g$, $\forall n \in \mathbb{Z}$, $\forall g \in \mathbb{F}_2$ is amenable and proper.

2. Except for the non properness statement, our result is contained in a result of Woess (see Theorem 1 in [6]): if $X = (V, E)$ is a locally finite graph and G admits an amenable action on V, then either G fixes a nonempty finite subset of V, or G fixes an end of X, or G fixes a unique pair of ends which are the fixed points of some hyperbolic element in G.

3. There are results on strong isoperimetric inequalities for graphs with infinitely many ends satisfying extra conditions (see Theorem 10.10 in [8]): these give alternative answers to Ceccherini's question.

4. A stronger question is to prove without appealing to Stallings' result that a finitely generated group with infinitely many ends, contains a free group on two generators. Such constructions can be found in the work of Woess (Theorem 3 in [7]), Karlsson and Noskov (Proposition 3 in [3]), and Karlsson (Theorem 1 in [2]).

5. For a finitely generated group with infinitely many ends, Abels shows, using Stallings' theorem, that for G a finitely generated group with infinitely many ends, the compact set of ends is actually a minimal G-space (Theorem 1 in [1]). This is false for compactly generated, non discrete groups. Abels indeed gives the example of the group of affine mappings $(x \mapsto ax + b)$ over $\mathbb{Q}_p$. This group G is $HNN(K, K, \varphi)$, where K is the group of affine mappings over $\mathbb{Z}_p$ and $\varphi : K \to K$ is given by $(x \mapsto ax + b) \mapsto (x \mapsto ax + pb)$. So G has infinitely many ends, but has a unique fixed point on its space of ends[1], which is therefore not G-minimal.

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2 Proof of the theorem

Let X be a countable, discrete set. A *compactification* of X is a compact space $\overline{X} = X \cup \partial X$ in which X is an open dense subset. If G is a group of permutations of X, we say that $\overline{X}$ is a *G-compactification* if the action of G on X extends to an action of G on $\overline{X}$ by homeomorphisms. When X is a locally finite graph (identified with its set of vertices), we will take for ∂X the set of ends of X. In this case, we say that $\overline{X} = X \cup \partial X$ is the *end-compactification* of X (it is an $Aut(X)$-compactification).

Lemma 2. *Assume that G admits an amenable action without finite orbit, on a countable set X. Let μ be G-invariant mean on X. Let $\overline{X}$ be a G-compactification of X. Then for every subset A of X with $\mu(A) = 1$, the set $\left(\bigcap_{g\in G} \overline{gA}\right) \cap \partial X$ is not empty.*

Proof. By compactness of ∂X, it is enough to show that the family $\left(\overline{gA} \cap \partial X\right)_{g\in G}$ has the finite intersection property. For $g_1, \ldots, g_n \in G$, we have $\mu\left(\bigcap_{i=1}^n g_i A\right) = 1$, while $\mu(F) = 0$ for every finite subset $F \subset X$ since G has no finite orbit. So $\bigcap_{i=1}^n g_i A$ is infinite. Therefore $\left(\overline{\bigcap_{i=1}^n g_i A}\right) \cap \partial X \neq \emptyset$. A fortiori $\bigcap_{i=1}^n \left(\overline{g_i A} \cap \partial X\right) \neq \emptyset$. $\square$

[1] This can be seen directly; it also follows from our result, as G is amenable as a discrete group.

The proof of Theorem 1 will follow from the four claims below:

Claim 1. Let K be a finite, connected subgraph of X. Let A be an unbounded connected component of $X \setminus K$. Then $gK \subset A$ for infinitely many g's in G.

By the assumption, any G-orbit in X has infinite intersection with A (indeed, the assumption implies that Gx is dense in $\overline{X}$ for every vertex x in V since G acts by isometries on X; therefore the intersection of Gx and A is infinite since $\overline{A}$ is a neighborhood of all ends contained in it). So for $x \in K$, one finds a sequence $(g_n)_{n\geq 1}$ in G such that $g_n x$'s are pairwise distinct vertices in A. Since $d(g_n x, x) \to \infty$ for $n \to \infty$, we have $g_n K \cap K = \emptyset$ for n sufficiently large. Then $g_n K$ is a connected subset of $X \setminus K$, and $g_n K \cap A \neq \emptyset$. By maximality of A among connected subsets of $X \setminus K$, this implies $g_n K \subset A$.

If K is a finite connected subgraph of X, we will say that K is *good* if every connected component of $X \setminus K$ is infinite. Let K be an arbitrary finite connected subgraph of X. Denote by $\widehat{K}$ the union of K and the finite connected components of $X \setminus K$; then $\widehat{K}$ is a good subgraph of X.

Claim 2. Let K be a good subgraph of X, such that $X \setminus K$ has at least 3 connected components. Let μ be G-invariant mean on V. Then there exists a unique connected component C_K of $X \setminus K$ such that $\mu(C_K) = 1$.

Indeed, let $A_1, \ldots, A_n$ be the connected components of $X \setminus K$ with $n \geq 3$. Without loss of generality, we may assume that $\mu(A_1) \leq \mu(A_i)$, $\forall i \in \{1, \ldots, n\}$. By claim 1, we can find $h \in A_1$ such that $hK \cap K = \emptyset$ and $hK \subset A_1$. Since $hA_1, \ldots, hA_n$ are the connected components of $X \setminus hK$, and K is connected, there exists a unique $k \in \{1, \ldots, n\}$ such that $K \subset hA_k$, so that $hA_i \subset A_1$, $\forall i \neq k$. Hence $\bigsqcup_{i\neq k} hA_i \subset A_1$ (see figure 1).

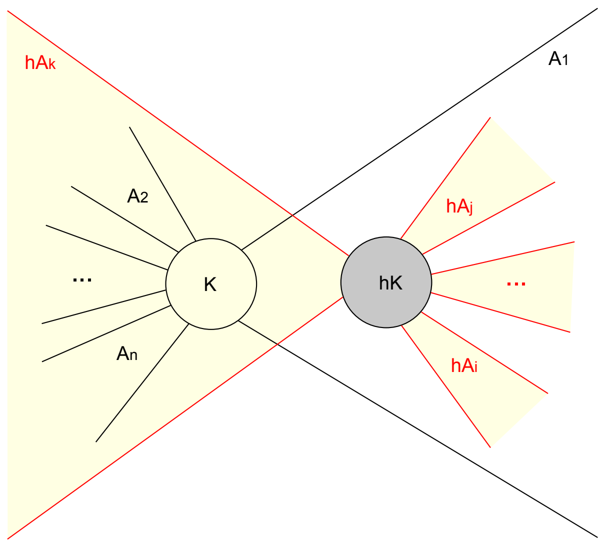

FIGURE 1.

Then by minimality of $\mu(A_1)$,

$$(n-1)\mu(A_1) \leq \sum_{i\neq k} \mu(A_i) = \sum_{i\neq k} \mu(hA_i) = \mu\big(\bigsqcup_{i\neq k} hA_i\big) \leq \mu(A_1).$$

Hence $\mu(A_1) = 0$ since $n \geq 3$, and $\mu(A_i) = 0$, $\forall i \neq k$. Since μ is zero on finite subsets of X, we have $1 = \mu(X) = \mu\big(K \cup \bigsqcup_{j=1}^{n} A_j\big) = \mu(A_k)$. We set $A_k = C_K$.

Let x_0 be a base-vertex in V. Denote by B_N the ball of radius N centered at x_0. Let N_0 be such that, for $N \geq N_0$, the complement $X \setminus \widehat{B_N}$ has at least 3 connected components. Set

$$D_N = \Big(\bigcap_{g\in G} \overline{gC_{\widehat{B_N}}}\Big) \cap \partial X.$$

By Lemma 2, $D_N \neq \emptyset$, and $(D_N)_{N\geq N_0}$ form a decreasing family of closed non-empty subsets of ∂X. So by compactness, $E = \bigcap_{N\geq N_0} D_N$ is non-empty, and obviously G-invariant.

Claim 3. The set E is reduced to one point, and G has no other fixed point in ∂X.

Indeed, if $w \in E$ and $w' \in \partial X$ with $w \neq w'$, then for N large enough w and w' are not in the same closure of connected component of $X \setminus \widehat{B_N}$. So $w \in \overline{C_{\widehat{B_N}}}$ and $w' \notin \overline{C_{\widehat{B_N}}}$, which means $w' \notin E$.

Let us show that $gw' \neq w'$ for a suitable $g \in G$. Recall (see e.g. Theorem 4 and 9 in [5]) that an automorphism $h \in Aut(X)$ is of exactly one of 3 possible types:

- elliptic, if h stabilizes some finite subset of V.
- parabolic, if h is non-elliptic and fixes exactly one end.
- hyperbolic, if h is non-elliptic and fixes exactly two ends.

Let $A' \neq C_{\widehat{B_N}}$ be a connected component of $X \setminus \widehat{B_N}$ with $w' \in \overline{A'}$. Let A be a connected component of $X \setminus \widehat{B_N}$ distinct from A' and $C_{\widehat{B_N}}$. By claim 1, we can find $g \in G$ such that $gB_N \subset A$. All connected components of $X \setminus \widehat{B_N}$ will be mapped into A by g, except one. This exceptional connected component is necessarily $C_{\widehat{B_N}}$ because $\mu(C_{\widehat{B_N}}) = 1$ and μ is G-invariant. In particular, $gA \subset A$, and this inclusion is strict. So $g^m A \subset A$, $\forall m \geq 1$. The sequence $g^m x_0$ possesses a subsequence $g^{m_k} x_0$ which converges to an end ξ in $\overline{A}$. It is obvious that g fixes ξ; therefore g is hyperbolic fixing exactly ξ and w. In particular, $gw' \neq w'$, as was to be shown.

Claim 4. The action of G (endowed with the discrete topology) on V is not proper.

The proof is inspired by a nice observation due to Karlsson and Noskov (Proposition 4 in [3]; see also Proposition 5 in [2]). As in claim 3, we can find $h \in G$ such that $h^m A' \subset A'$, $\forall m \geq 1$ so that h is hyperbolic and fixes exactly one end η in $\overline{A'}$, apart from w. With the same g as in Claim 3, let $y_n = h^n g h^{-n}$. We claim that $y_n \neq y_m$, $\forall n \neq m$. Suppose by contradiction that there is $n \neq m$ such that $h^n g h^{-n} = h^m g h^{-m}$; so there exists $k \neq 0$ such that $h^k g = g h^k$. Then

$h^k g\eta = gh^k\eta = g\eta$ since h fixes η. Since h^k fixes the same ends as h, $g\eta$ has to be η or w. But this is not possible since η, ξ and w are all distinct.

Now, it remains for us to prove that the set $\{y_n x_0 : n \in \mathbb{N}\}$ is bounded. Indeed, for γ a hyperbolic automorphism, let $\ell(\gamma) =: \min\{d(\gamma^k v, v) : k \in \mathbb{Z}\backslash\{0\}, v \in V\}$ be the translation length of γ, and let $L_\gamma =: \{v \in V : d(\gamma v, v) = \ell(\gamma)\}$ be the axis of γ (this is a line in X). We will use one more result of Halin [5]: the end w, being a fixed end of some hyperbolic automorphism, is *thin*, i.e. for $N \gg 1$ the set C_{B_N} contains finitely many disjoint rays. As a consequence, the rays $L_h \cap C_{\widehat{B_N}}$ and $L_g \cap C_{\widehat{B_N}}$ stay within finite distance, i.e. there exists $R > 0$ such that, for every $x \in L_h \cap C_{\widehat{B_N}}$, one can find $x' \in L_g \cap C_{\widehat{B_N}}$ with $d(x, x') \leq R$.

To prove that $\{y_n x_0 : n \in \mathbb{N}\}$ is bounded, we may clearly assume that $x_0 \in L_h$. For n large enough, we have $h^{-n}x_0 \in C_{B_N}$, so we can find $x_n \in L_g$ with $d(h^{-n}x_0, x_n) \leq R$. Then,

$$\begin{aligned} d(y_n x_0, x_0) &= d(gh^{-n}x_0, h^{-n}x_0) \\ &\leq d(gh^{-n}x_0, gx_n) + d(gx_n, x_n) + d(x_n, h^{-n}x_0) \\ &\leq 2R + \ell(g); \end{aligned}$$

this concludes the proof.

References

[1] H. Abels, *On a problem of Freudenthal's*, Compositio Mathematica, **35** (1977), no. 1, 39-47.

[2] A. Karlsson, *Free subgroups of groups with nontrivial Floyd boundary*, Comm. Algebra, **31** (2003) 5361-5376.

[3] A. Karlsson and Guennadi A. Noskov, *Some groups having only elementary actions on metric spaces with hyperbolic boundaries*, Geom. Dedicata, **104** (2004) 119-137.

[4] J. R. Stallings, *On torsion-free groups with infinitely many ends*, Ann. of Math. (2) **88** (1968) 312-334.

[5] R. Halin, *Automorphisms and endomorphisms of infinite locally finite graphs*, Abh. Math. Sem. Univ. Hamburg. **39** (1973), 251-283.

[6] W. Woess, *Amenable group actions on infinite graphs*, Math. Ann. **284**, (1989), 251-265.

[7] W. Woess, *Fixed sets and free subgroups of groups acting on metric spaces*, Math. Z. **214** (1993), no. 3, 425-439.

[8] W. Woess, *Random walks on infinite graphs and groups*, Cambridge tracts in mathematics 138, Cambridge university press, 2000.

Institut de Mathématiques - Université de Neuchâtel
11, Rue Emile Argand - BP 158, 2009 Neuchâtel - Switzerland
E-mail: so.moon@unine.ch, alain.valette@unine.ch