Proper actions of lamplighter groups associated with free groups
(Actions propres du groupe de l'allumeur de réverbères associé au groupe libre)

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Abstract

Given a finite group H and a free group $\mathbf{F}_n$, we prove that the wreath product $H \wr \mathbf{F}_n$ admits a metrically proper, isometric action on a Hilbert space.

Resumé en français

Soit H un groupe fini et F un groupe libre, ou plus généralement un groupe admettant une structure d'espace à murs invariante à gauche et propre. Nous montrons que le produit en couronne $H \wr F = H^{(F)} \rtimes F$ admet également une telle structure d'espace à murs. En conséquence, il a la propriété de Haagerup, c'est-à-dire qu'il possède une action isométrique métriquement propre sur un espace de Hilbert.

1   Introduction

Following [HP98], a *space with walls* is a pair $(X, \mathcal{W})$ where X is a set and $\mathcal{W}$ is a family of partitions of X into two classes, called *walls*, such that for any two distinct points $x, y \in X$, the number $w(x, y)$ of walls separating x from y, called the wall distance between x and y, is finite.

Let us define the class $\mathcal{PW}$ as the class of countable groups G admitting a left-invariant structure of space with walls such that the corresponding wall

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distance is proper, i.e. bounded subsets are finite. The class $\mathcal{PW}$ contains for instance $\mathbf{Z}^n$, free groups, surface groups (see 1.2.7 in [CCJ$^+$01] for this fact). More generally, it contains all groups acting combinatorially properly on some finite product of trees.

It is known (see [CCJ$^+$01], Corollary 7.4.2) that, if G belongs to the class $\mathcal{PW}$, then G has the *Haagerup property* (or is *a-T-menable*), i.e. G admits an isometric action on a Hilbert space $\mathcal{H}$ that is metrically proper, that is,

$$\lim_{g \to \infty} \|x_0 - g \cdot x_0\| = +\infty$$

for some/every x_0 in $\mathcal{H}$.

Recall that the (standard, restricted) *wreath product $H \wr G$* of two groups H and G is the semidirect product $H^{(G)} \rtimes G$, where G acts by shifting the direct sum $H^{(G)}$ of copies of H. Up to now, the problem of stability of the Haagerup Property under wreath products was open. Indeed, the Haagerup Property is closed under direct sums, but not under general semidirect products. However it is known to be closed under extensions with *amenable* quotients [CCJ$^+$01, Example 6.1.6], so that in particular $H \wr G$ is Haagerup whenever H is Haagerup and G is amenable.

These were however the only known examples of wreath products with the Haagerup Property. This note presents the first examples of a-T-menable wreath products $H \wr G$ with $H \neq 1$ and G non-amenable, including the first natural such example, namely the "lamplighter group" $(\mathbf{Z}/2\mathbf{Z}) \wr \mathbf{F}_2$ over the free group $\mathbf{F}_2$ of rank two.

Theorem 1 *Let H be a finite group. If G is a group in $\mathcal{PW}$, then so is $H \wr G$. In particular, $H \wr G$ has the Haagerup property.*

This latter statement will be generalized in a forthcoming paper, where we prove that the Haagerup Property is closed under taking wreath products. The proof of the general case relies on the same basic construction, but requires more technicalities.

Theorem 1 is proved in Section 2, while Section 3 presents some consequences on the relation between the Haagerup Property and weak amenability.

Frédéric Haglund for very fruitful discussions and hints about the construction of families of walls for wreath products.

2   Proof of Theorem 1

We first fix some notation. Write $\Lambda = H^{(G)}$ and $\Gamma = H \wr G = \Lambda \rtimes G$. Elements of Γ are denoted

$$\gamma = \lambda g \ (\lambda \in \Lambda, g \in G).$$

The *support* of λ is

$$\mathrm{supp}(\lambda) = \{g \in G : \lambda(g) \neq 1_H\}.$$

If $(X, \mathcal{W})$ is a space with walls, a *half-space* in X is one of the two classes of some wall in $\mathcal{W}$. Suppose that G belongs to the class $\mathcal{PW}$, and let us denote by $\mathcal{H}$ the family[1] of half-spaces in G. For $A \in \mathcal{H}$, we denote by A^c the other half-space in the same wall, i.e. the complement of A in G. For $A \in \mathcal{H}$ and $\mu : A^c \to H$ a function with finite support, we set

$$E(A, \mu) =: \{\gamma = \lambda g \in \Gamma : g \in A, \ \lambda|_{A^c} = \mu\}.$$

We define a family of walls in Γ as partitions $\{E(A, \mu), E(A, \mu)^c\}$, for $A \in \mathcal{H}$ and $\mu : A^c \to H$ finitely supported. We check in three steps that this equips Γ with a structure of space with walls on which Γ acts properly.

<u>1st step</u>: Γ is a space with walls.

Let $\gamma_1 = \lambda_1 g_1$ and $\gamma_2 = \lambda_2 g_2$ be two elements of Γ. Let us show that there are finitely many $E(A, \mu)$'s such that $\gamma_1 \in E(A, \mu)$ and $\gamma_2 \notin E(A, \mu)$.

Indeed $\gamma_1 \in E(A, \mu)$ means $g_1 \in A$ and $\lambda_1|_{A^c} = \mu$ (so that μ is determined once A is given). And $\gamma_2 \notin E(A, \mu)$ means that either $g_2 \notin A$, or $\lambda_2|_{A^c} \neq \mu$; since $\mu = \lambda_1|_{A^c}$, this can be re-written:

$$A^c \cap (\{g_2\} \cup \mathrm{supp}(\lambda_1^{-1}\lambda_2)) \neq \emptyset.$$

So A must separate g_1 from the finite set $\{g_2\} \cup \mathrm{supp}(\lambda_1^{-1}\lambda_2)$. Since G is a space with walls, this singles out finitely many possibilities for a.

<u>2nd step</u>: Γ preserves the above wall structure.

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[1] Although the family is not assumed injective, we will identify, to avoid subscripts in the notation, elements of the index set $\mathcal{H}$ with the corresponding subsets of G.

This follows immediately from the formulae:

$$gE(A, \mu) = E(gA, g\mu) \ \ (g \in G);$$

$$\lambda E(A, \mu) = E(a, \lambda|_{A^c}\mu) \ \ (\lambda \in \Lambda).$$

3rd step: Γ acts metrically properly on its wall structure.

Let $w_\Gamma(\gamma)$ be the number of walls separating the unit 1_Γ from $\gamma \in \Gamma$. We must prove that, for every $N \in \mathbf{N}$, there are finitely many γ's with $w_\Gamma(\gamma) \leq N$.

Define analogously $w_G(g)$ as the number of walls separating 1_G from g in G, and set $B_G(N) = \{g \in G : w_G(g) \leq N\}$; by our assumption $B_G(N)$ is a finite set.

Claim: If $\gamma = \lambda g$ satisfies $w_\Gamma(\gamma) \leq N$, then $\{g\} \cup \mathrm{supp}(\lambda) \subset B_G(N)$.

Theorem 1 then follows from the claim together with the fact that H is a finite group.

Proof of the claim: Contraposing, suppose that there exists $g' \in \{g\} \cup \mathrm{supp}(\lambda)$ with $w_G(g') > N$. So we find $N + 1$ distinct half-spaces $A_0, ..., A_N$ in $\mathcal{H}$ with $1_G \in A_i$ and $g' \notin A_i$ $(i = 0, ..., N)$. Then the $E(A_i, 1_\Lambda)$'s are $N + 1$ distinct half-spaces in Γ separating 1_Γ from $\gamma = \lambda g$, so $w_\Gamma(\gamma) > N$. $\qquad\square$

3 Weak amenability à la Cowling-Haagerup

Theorem 1 has interesting consequences in view of a recent result of Ozawa and Popa [OP]. Recall from [CH89] that a countable group G is *weakly amenable* if there exists a constant $L > 0$ and a sequence $(f_n)_{n>0}$ of functions with finite support on G, converging pointwise to 1, and such that $\|f_n\|_{cb} \leq L$ for $n > 0$, where $\|f\|_{cb}$ is the Herz-Schur multiplier norm of the function f. The best (i.e. lowest) possible L for which there exists such a sequence is the *Cowling-Haagerup constant* of G, denoted by $\Lambda(G)$. We set $\Lambda(G) = \infty$ if G is not weakly amenable. Groups which are weakly amenable with constant 1 are also said to satisfy the *complete metric approximation property*.

In [OP, Corollary 2.11], it is proved that, if H is non-trivial and G is non-amenable, then $H \wr G$ does not have the complete metric approximation approximation property. Combining with Theorem 1, we get:

Corollary 2 *For H a non-trivial finite group, $H \wr \mathbf{F}_2$ is an a-T-menable group without the complete metric approximation property.* $\qquad\square$

This disproves a conjecture of Cowling (see page 7 in [CCJ$^+$01]), stating that the class of a-T-menable groups coincides with the class of groups with the complete metric approximation property. Whether every such group is a-T-menable, is still an open question.

It was pointed out to us by N. Ozawa that from Corollary 2 one can deduce the following:

Corollary 3 *Let H be a non-trivial finite group. The iterated wreath product $(H \wr \mathbf{F_2}) \wr \mathbf{Z}$ is a-T-menable but not weakly amenable.*

Proof: It was already observed above that $G =: (H \wr \mathbf{F_2}) \wr \mathbf{Z}$ is a-T-menable; on the other hand, for $N \geq 1$ consider a subgroup K_N of G which is the direct sum of N copies of $H \wr \mathbf{F_2}$. Then by Proposition 1.3.(a) and Corollary 1.5 in [CH89], we have $\Lambda(G) \geq \Lambda(K_N) = \Lambda(H \wr \mathbf{F_2})^N$. Since N is arbitrary and $\Lambda(H \wr \mathbf{F_2}) > 1$ by the Ozawa-Popa result [OP], we get $\Lambda(G) = \infty$. $\square$

In view of the cubulation of spaces with walls, carried out independently in [CN05] and [Nic04], we get from Theorem 1:

Corollary 4 *Let H be a non-trivial finite group. The wreath product $H \wr \mathbf{F}_2$ admits a metrically proper, isometric action on a $CAT(0)$ cube complex, but does not have the complete metric approximation property.* $\square$

In contrast, it was recently proved by Guentner and Higson [GH] that a group acting metrically properly, isometrically on a *finite-dimensional $CAT(0)$* cube complex, has the complete metric approximation property.

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